



Data Structures and Algorithms

Probability in Computer Science

CS 225
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Department of Computer Science



Learning Objectives

Formalize the concept of randomized algorithms

Review fundamentals of probability in computing

Distinguish the three main types of 'random' in computer science

Randomized Algorithms

A **randomized algorithm** is one which uses a source of randomness somewhere in its implementation.

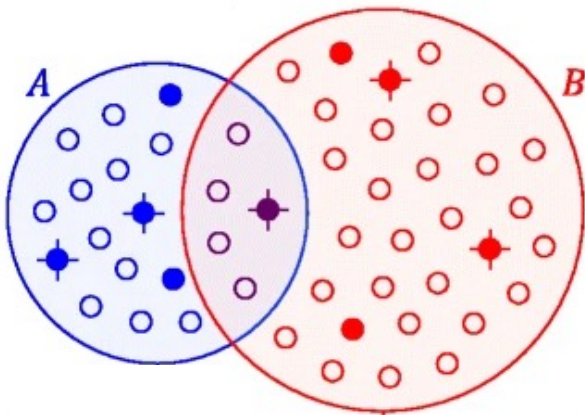
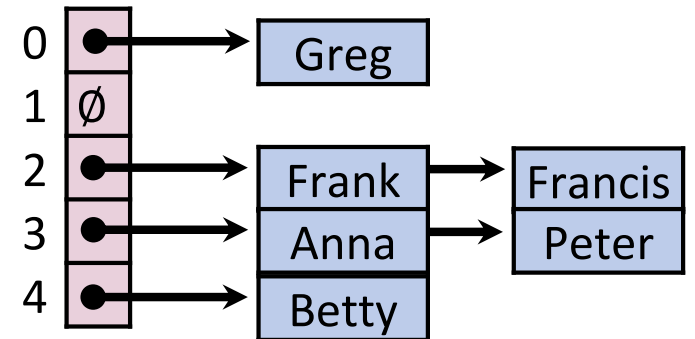
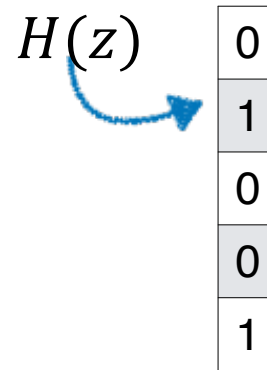


Figure from Ondov et al 2016



$H(x)$	0	2	1	0	0	4	0	2	0	6
$H(y)$	1	0	2	3	1	0	3	4	0	1
$H(z)$	2	1	0	2	0	1	0	0	7	2



Quick Primes with Fermat's Primality Test

If p is prime and a is not divisible by p , then $a^{p-1} \equiv 1 \pmod{p}$

But... ***sometimes*** if n is composite and $a^{n-1} \equiv 1 \pmod{n}$



Fundamentals of Probability

Imagine you roll a pair of six-sided dice.

The **sample space** Ω is the set of all possible outcomes.

An **event** $E \subseteq \Omega$ is any subset.



Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

A **random variable** is a function from events to numeric values.

The **expectation** of a (discrete) random variable is:

$$E[X] = \sum_{x \in \Omega} Pr\{X = x\} \cdot x$$



Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

$$E[X + Y] = ?$$



Fundamentals of Probability

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Linearity of Expectation: For any two random variables X and Y ,

$$E[X + Y] = E[X] + E[Y]$$

Fundamentals of Probability

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Linearity of Expectation: For any two random variables X and Y ,

$$E[X + Y] = E[X] + E[Y]$$

$$| = \sum_x \sum_y (x + y) Pr\{X = x, Y = y\}$$

$$| = \sum_x x \sum_y Pr\{X = x, Y = y\} + \sum_y y \sum_x Pr\{X = x, Y = y\}$$

$$| = \sum_x x \cdot Pr\{X = x\} + \sum_y y \cdot Pr\{Y = y\}$$

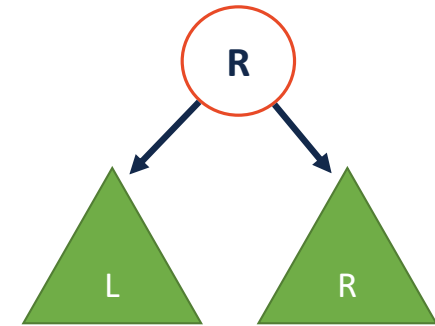


Randomization in Algorithms

1. Assume input data is random to estimate average-case performance
2. Use randomness inside algorithm to estimate expected running time
3. Use randomness inside algorithm to approximate solution in fixed time

Average-Case Analysis: BST

Smallest Largest





Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Claim: $S(n)$ is $O(n \log n)$

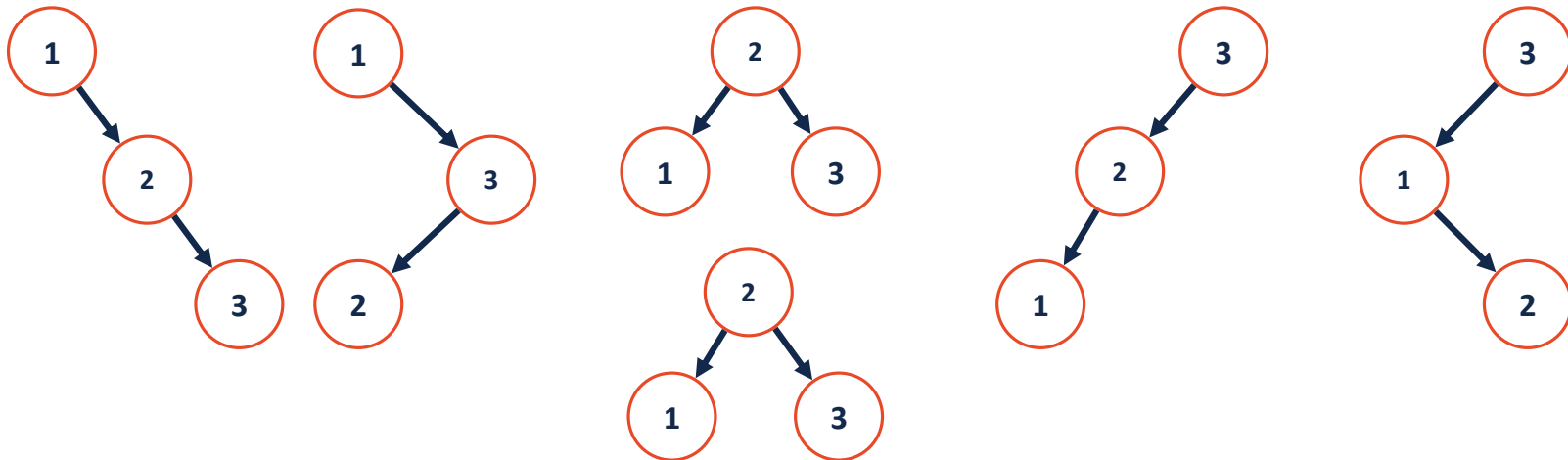
N=0:

N=1:

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

N=3:





Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

IH for all $0 \leq k < n$ $S(k)$ is $O(k \log k)$



Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Let $0 \leq i \leq n - 1$ be the number of nodes in the left subtree.

Then for a fixed i , $S(n) = (n - 1) + S(i) + S(n - i - 1)$



Average-Case Analysis: BST

Let $S(n)$ be the **average** total internal path length **over all BSTs** that can be constructed by uniform random insertion of n objects

$$S(n) = (n - 1) + \frac{1}{n} \sum_{i=1}^{n-1} S(i) + S(n - i - 1)$$

Average-Case Analysis: BST

$$S(n) = (n - 1) + \frac{2}{n} \sum_{i=1}^{n-1} S(i)$$

$$S(n) = (n - 1) + \frac{2}{n} \sum_{i=1}^{n-1} (ci \ln i)$$

$$S(n) \leq (n - 1) + \frac{2}{n} \int_1^n (cx \ln x) dx$$

$$S(n) \leq (n - 1) + \frac{2}{n} \left(\frac{cn^2}{2} \ln n - \frac{cn^2}{4} + \frac{c}{4} \right) \approx cn \ln n$$



Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects. Since $S(n)$ is $O(n \log n)$, if we assume we are randomly choosing a node to insert, find, or delete* then each operation takes:



Average-Case Analysis: BST

Summary: All operations are on average $O(\log n)$

Randomness:

Assumptions:

Expectation Analysis: Randomized Quicksort

6	1	0	3	7	9	2	4
---	---	---	---	---	---	---	---

1	0	3	2	4	9	6	7
---	---	---	---	---	---	---	---

1	0	3	2	4	9	6	7
---	---	---	---	---	---	---	---

1	0	2	3	4	6	7	9
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Expectation Analysis: Randomized Quicksort

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0	1	2	3	4	5	6	7
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...

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Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected time is $O(n \log n)$ *for any input!*

Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

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Let X be the total comparisons and X_{ij} be an **indicator variable**:

$$X_{ij} = \begin{cases} 1 & \text{if } i\text{th object compared to } j\text{th} \\ 0 & \text{if } i \text{ object not compared to } j\text{th} \end{cases}$$

Then...



Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \frac{2}{j-i+1}$.

Base Case: (N=2)

Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \frac{2}{j-i+1}$ **Induction:** Assume true for all inputs of $< n$





Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} E[X_{ij}] \quad E[X_{ij}] = \frac{2}{j-i+1}$$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^n \sum_{j=i+1}^n E[X_{ij}] \quad E[X_{ij}] = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^n 2 \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-i+1} \right)$$

$$E[X] = \sum_{i=1}^n 2(H_{n-1} - 1) \leq 2n \cdot H_n \leq 2n \ln n$$



Expectation Analysis: Randomized Quicksort

Summary: Randomized quick sort is $O(n \log n)$ regardless of input

Randomness:

Assumptions: