March 20 – Disjoint Sets
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Disjoint Sets

- Set 1: 2, 5, 9
- Set 2: 7
- Set 3: 0, 1, 4, 8
- Set 4: 3, 6
Disjoint Sets

Operation: find(4)
Disjoint Sets

Operation: $\text{find}(4) == \text{find}(8)$
Disjoint Sets

Key Ideas:
• Each element exists in exactly one set.
• Every set is an equitant representation.
  • Mathematically: $4 \in [0]_R \rightarrow 8 \in [0]_R$
  • Programmatically: `find(4) == find(8)`
Disjoint Sets ADT

• Maintain a collection $S = \{s_0, s_1, \ldots, s_k\}$

• Each set has a representative member.

• API:  
  void makeSets(int number);
  void union(int k1, const int k2);
  int find(int k);
Implementation #1

Find(k):

Union(k1, k2):
YOU EXPECTED A NEW DATA STRUCTURE

BUT IT WAS ME, TREE ALL ALONG
Implementation #2

Find(k):

Union(k1, k2):
Implementation #2

- We will continue to use an array where the index is the key.

- The value of the array is:
  - -1, if we have found the representative element
  - The index of the parent, if we haven’t found the rep. element

- We will call these UpTrees:
### UpTrees

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- Diagram showing four trees with nodes labeled 0 to 3.
Disjoint Sets

Find(k):

Union(k1, k2):
Disjoint Sets Find

```cpp
int DisjointSets::find(int i) {
    if ( s[i] < 0 ) { return i; }
    else { return find( s[i] ); }
}
```

Running time?

What is the ideal UpTree?
Disjoint Sets Union

```cpp
void DisjointSets::union(int r1, int r2) {
```

![Diagram showing the disjoint sets union process]
Disjoint Sets – Union

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Disjoint Sets – Smart Union

**Union by height**

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**Idea:** Keep the height of the tree as small as possible.
Disjoint Sets – Smart Union

**Union by size**

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*Idea:* Keep the height of the tree as small as possible.
Disjoint Sets – Smart Union

**Union by height**

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**Idea**:
- Keep the height of the tree as small as possible.
- Minimize the number of nodes that increase in height.

Both guarantee the height of the tree is: ______________.
Disjoint Sets Find

```cpp
int DisjointSets::find(int i) {
    if ( s[i] < 0 ) { return i; }
    else { return find( s[i] ); }
}
```

```cpp
void DisjointSets::unionBySize(int root1, int root2) {
    int newSize = arr_[root1] + arr_[root2];

    // If arr_[root1] is less than (more negative), it is the larger set;
    // we union the smaller set, root2, with root1.
    if ( arr_[root1] < arr_[root2] ) {
        arr_[root2] = root1;
        arr_[root1] = newSize;
    }

    // Otherwise, do the opposite:
    else {
        arr_[root1] = root2;
        arr_[root2] = newSize;
    }
```
Path Compression
Disjoint Sets Analysis

The **iterated log** function:

*The number of times you can take a log of a number.*

\[
\log^*(n) = \\
0 \quad , \quad n \leq 1 \\
1 + \log^*(\log(n)) \quad , \quad n > 1
\]

What is \( \lg^*(2^{65536}) \)?
Disjoint Sets Analysis

In a Disjoint Sets implemented with smart **unions** and path compression on **find**:

Any sequence of **m union** and **find** operations result in the worse case running time of $O(\text{__________} )$, where $n$ is the number of items in the Disjoint Sets.