November 4 – Disjoint Sets

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Disjoint Sets
Disjoint Sets

Operation: find(4)
Disjoint Sets

Operation: \texttt{find(4) == find(8)}
Disjoint Sets

Operation:
if ( find(2) != find(7) ) {
    union( find(2), find(7) );
}
Disjoint Sets

Key Ideas:
• Each element exists in exactly one set.
• Every set is an equitant representation.
  • Mathematically: \(4 \in [0]_R \implies 8 \in [0]_R\)
  • Programmatically: \(\text{find}(4) == \text{find}(8)\)
Disjoint Sets ADT

• Maintain a collection $S = \{s_0, s_1, \ldots s_k\}$

• Each set has a representative member.

• API: 
  void makeSet(const T & t);
  void union(const T & k1, const T & k2);
  T & find(const T & k);
Implementation #1

Find(k):

Union(k1, k2):
Implementation #2

• We will continue to use an array where the index is the key

• The value of the array is:
  • -1, if we have found the representative element
  • The index of the parent, if we haven’t found the rep. element

• We will call theses UpTrees:

```
  0 1 2 3  
0 -1 -1 -1 -1
-1  -1  -1  -1
```

```
UpTrees
Disjoint Sets

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<thead>
<tr>
<th></th>
<th>0</th>
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<th>3</th>
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<tr>
<td>0</td>
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<td>6</td>
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</tbody>
</table>
Disjoint Sets Find

```cpp
int DisjointSets::find(int i) {
    if ( s[i] < 0 ) { return i; } else { return _find( s[i] ); } }
```

Running time?

What is the ideal UpTree?
Disjoint Sets Union

```cpp
void DisjointSets::union(int r1, int r2) {
}
```
Disjoint Sets – Union

0 1 2 3 4 5 6 7 8 9 10 11
6 6 6 8 -1 10 7 -1 7 7 4 5
Disjoint Sets – Smart Union

**Union by height**

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<thead>
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*Idea:* Keep the height of the tree as small as possible.
Disjoint Sets – Smart Union

Idea: Keep the height of the tree as small as possible.

Idea: Minimize the number of nodes that increase in height

Both guarantee the height of the tree is: _______________.
Disjoint Sets Find

```cpp
int DisjointSets::find(int i) {
    if ( s[i] < 0 ) { return i; }  
    else { return _find( s[i] ); } 
}
```

```cpp
void DisjointSets::unionBySize(int root1, int root2) {
    int newSize = arr_[root1] + arr_[root2];
    // If arr_[root1] is less than (more negative), it is the larger set;
    // we union the smaller set, root2, with root1.
    if ( arr_[root1] < arr_[root2] ) {
        arr_[root2] = root1;
        arr_[root1] = newSize;
    }  
    else {  
        arr_[root1] = root2;
        arr_[root2] = newSize;
    }
}
```
Path Compression
Disjoint Sets Analysis

The **iterated log** function:

*The number of times you can take a log of a number.*

\[
\log^*(n) = \\
0, \quad n \leq 1 \\
1 + \log^*(\log(n)), \quad n > 1
\]

What is \(\lg^*(2^{65536})\)?
Disjoint Sets Analysis

In an Disjoint Sets implemented with smart \texttt{unions} and path compression on \texttt{find}:

Any sequence of \texttt{m union} and \texttt{find} operations result in the worse case running time of $O( \text{___________} )$, where $n$ is the number of items in the Disjoint Sets.