Data Structures

MST3 & Single Source Shortest Path

CS 225

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Announcements

No graph traversal

Exam 4 this week!

MP_Puzzle releases today! **(Last MP)**

All deadlines which would end during fall break due AFTER break

Learning Objectives

Finish discussion about Prim's runtime

Compare Kruskal and Prim MST Algorithms

Introduce Single-Source Shortest Path Problem

Discuss Dijkstra's Algorithm

Extend to All-Paths Shortest Path (if time)

Kruskal's Algorithm

$|V| = n, |E| = m$

Priority Queue:	Total Running Time
Heap	O(n) + O(m) + O(m log n)
Sorted Array	O(n) + O(m log n) + O(m)

Global greedy

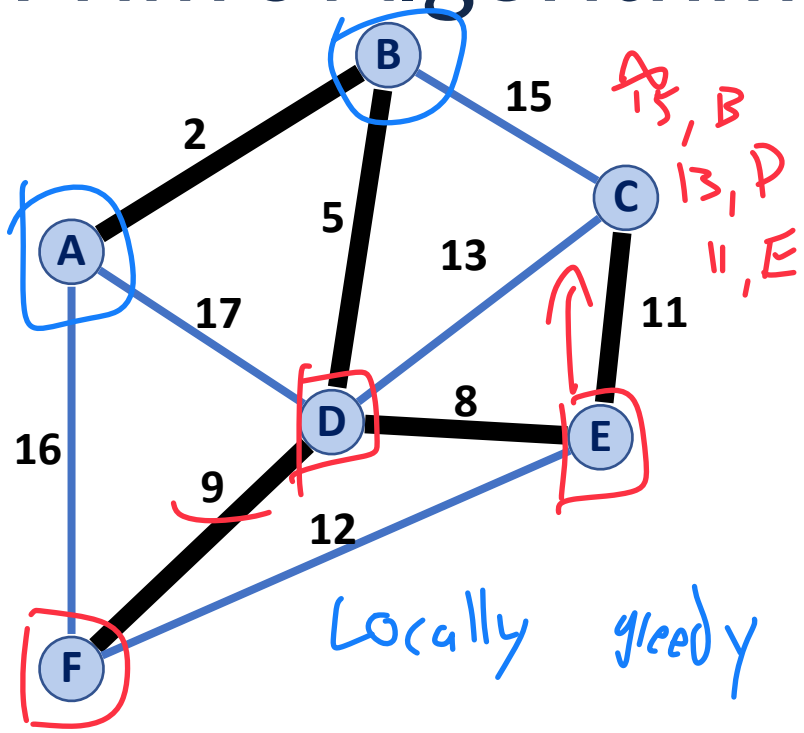
```
1  KruskalMST(G):
2    DisjointSets forest
3    foreach (Vertex v : G.vertices()):
4      forest.makeSet(v)
5
6    PriorityQueue Q     // min edge weight
7    Q.buildFromGraph(G.edges())
8
9    Graph T = (V, {})
10
11   while |T.edges()| < n-1:
12     Vertex (u, v) = Q.removeMin()
13     if forest.find(u) != forest.find(v):
14        T.addEdge(u, v)
15        forest.union( forest.find(u),
16                      forest.find(v) )
17
18   return T
19
```

All edge weights

Connect new items

Pick the best weights from small to large

Prim's Algorithm



A	B	C	D	E	F
0, —	2, A	11, E	5, B	8, D	9, D

```
1  PrimMST(G, s):
2    Input: G, Graph;
3           s, vertex in G, starting vertex
4    Output: T, a minimum spanning tree (MST) of G
5
6    foreach (Vertex v : G.vertices()):
7      d[v] = +inf
8      p[v] = NULL
9    d[s] = 0
10
11   PriorityQueue Q   // min distance, defined by d[v]
12   Q.buildHeap(G.vertices())
13   Graph T           // "labeled set"
14
15   repeat n times:
16     Vertex m = Q.removeMin()
17     T.add(m)
18     foreach (Vertex v : neighbors of m not in T):
19       if cost(v, m) < d[v]:
20         d[v] = cost(v, m)
21         p[v] = m
22
23   return T
```

Prim's Big O

$|V| = n, |E| = m$

```
 6  PrimMST(G, s):
 7    foreach (Vertex v : G.vertices()):
 8      d[v] = +inf
 9      p[v] = NULL
10    d[s] = 0
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12    PriorityQueue Q // min distance, defined by d[v]
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18      T.add(m)
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21          d[v] = cost(v, m)
22          p[v] = m
23
```

Init
O(n)

O(n) vertices

n×

remove min

check neighbors

update cost

12 - 14
- minheap: O(n)

compl. edvl.

21?

Our Priority Queue is built on distances

Prim's Big O

|V| = n, |E| = m

7 — 9: O(n)

12—14:

MinHeap: O(n)

Unsorted Array: O(1)

16—22: Complicated!

```
 6  PrimMST(G, s):
 7    foreach (Vertex v : G.vertices()):
 8      d[v] = +inf
 9      p[v] = NULL
10    d[s] = 0
11
12    PriorityQueue Q // min distance, defined by d[v]
13    Q.buildHeap(G.vertices())
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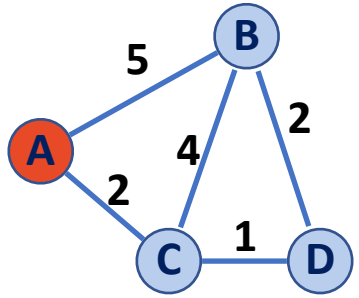
return G.vertices()

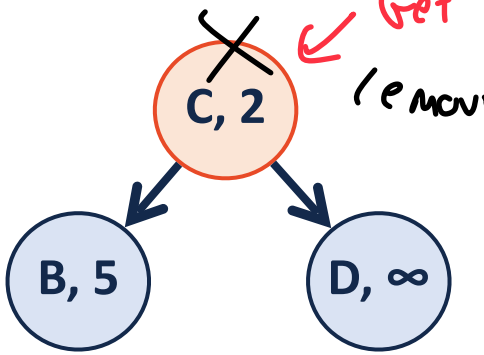
might not be O(1)

Depends on choice of **PriorityQueue** (MinHeap vs Unsorted Array)

Depends on choice of **Graph** (Adjacency Matrix vs Adjacency List)

A	B	C	D
0	5	2	∞





Get min in O(1)

remove min

Adjust our heap




```
 6  PrimMST(G, s):
 7    foreach (Vertex v : G.vertices()):
 8      d[v] = +inf
 9      p[v] = NULL
10    d[s] = 0
11
12    PriorityQueue Q // min distance, defined by d[v]
13    Q.buildHeap(G.vertices())
14    Graph T = new Graph(); // "labeled set"
15  for each vertex in order
16   repeat n times:
17     Vertex m = Q.removeMin()
18     T.add(m)
19     foreach (Vertex v : neighbors of m not in T):
20       if cost(v, m) < d[v]:
21         d[v] = cost(v, m)
22         p[v] = m
23
```

O(log n)

loop through neighbors

O(n) for AM

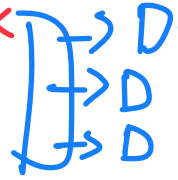
O(deg(v)) for AL



X O(1) O(1)

Always searches n spaces in array

n x (search a n size array)



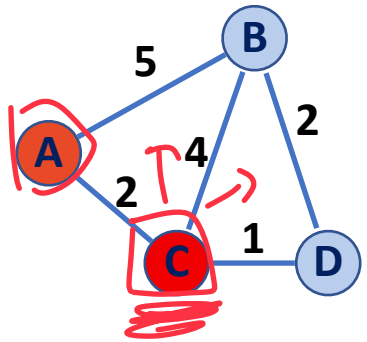
D→D each is deg(v) in length

Loop all vertices

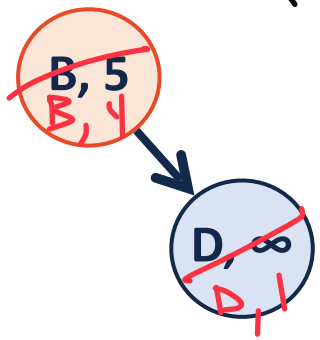
$\sum deg(v) = 2E = O(m)$

	Adj. Matrix	Adj. List
(min) Heap	O(n log n)	O(n log n)
	O(n) + _____ + O(n^2) + _____	O(n) + _____ + O(m) + _____

A	B	C	D
0	5, A	2, A	∞

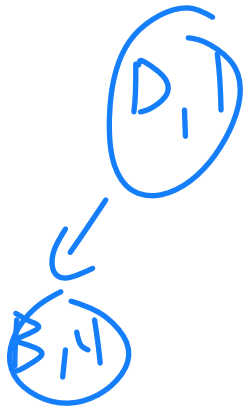


When we have heap we can assume updating value in array is O(1) time



Then we update our heap using heapify()

O(log n)




```
 6  PrimMST(G, s):
 7    foreach (Vertex v : G.vertices()):
 8      d[v] = +inf
 9      p[v] = NULL
10    d[s] = 0
11
12    PriorityQueue Q // min distance, defined by d[v]
13    Q.buildHeap(G.vertices())
14    Graph T          // "labeled set"
15
16    repeat n times:
17      Vertex m = Q.removeMin()
18      T.add(m)
19      foreach (Vertex v : neighbors of m not in T):
20        if cost(v, m) < d[v]:
21          d[v] = cost(v, m)
22          p[v] = m
23
```

$\sum_v \deg(v) = m$ — per vertex

O(log n) per edge

What is the Big O of line 21? (The remaining blank)

O(1) / O(log n) / O(log m) / O(n) / O(m)

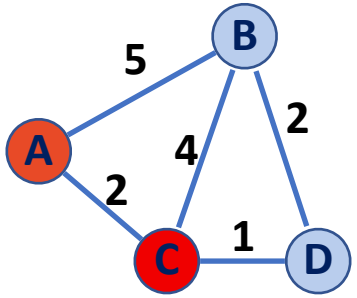
to update in array is this!

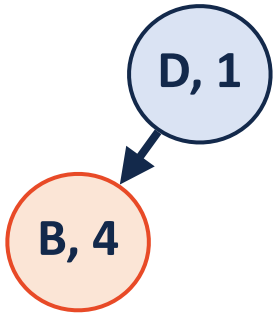
we store best (shortest) edge

we don't stop all edges

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + O(m log n)	O(n) + O(n log n) + O(m) + O(m log n)

A	B	C	D
0	4	2, A	1






```
 6  PrimMST(G, s):
 7    foreach (Vertex v : G.vertices()):
 8      d[v] = +inf
 9      p[v] = NULL
10    d[s] = 0
11
12    PriorityQueue Q // min distance, defined by d[v]
13    Q.buildHeap(G.vertices())
14    Graph T          // "labeled set"
15
16    repeat n times:
17      Vertex m = Q.removeMin()
18      T.add(m)
19      foreach (Vertex v : neighbors of m not in T):
20        if cost(v, m) < d[v]:
21          d[v] = cost(v, m)
22          p[v] = m
23
```

1) Change minheap value

2) HeapifyUp()

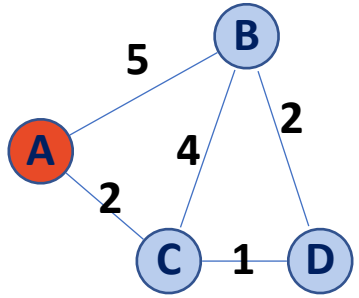
O(1)

O(log n)

line 21

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + O(m log n)	O(n) + O(n log n) + O(m) + O(m log n)

(A, 0)
(D, ∞)
(C, 2)
(B, 5)




```
 6  PrimMST(G, s):
 7    foreach (Vertex v : G.vertices()):
 8      d[v] = +inf
 9      p[v] = NULL
10    d[s] = 0
11
12    PriorityQueue Q // min distance, defined by d[v]
13    Q.buildHeap(G.vertices())
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20        if cost(v, m) < d[v]:
21          d[v] = cost(v, m)
22          p[v] = m
23
```

Unsorted array as PQ

O(n) to find to remove

O(1) ← $\sum_v$ deg(v) = m

b/c map we have pointer to address in array

What is the size of array?

O(n) one entry for vertex storing current best dist to vertex from any source

$n - 1 \leq m \leq O(n^2)$

	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$
Unsorted Array	$O(n^2)$	$O(n^2)$ ← only $O(n)$ & not $O(n^2 + m)$

Prim's Algorithm

Perfect example of bad tradeoff!

Sparse Graph: m ~ n

Heap + adj list

$O(n \log n)$

Dense Graph: m ~ n^2

If $m = n^2$ heap becomes $O(n^2 + n^2 \log n)$

```
 6  PrimMST(G, s):
 7    foreach (Vertex v : G.vertices()):
 8      d[v] = +inf
 9      p[v] = NULL
10    d[s] = 0
11
12    PriorityQueue Q // min distance, defined by d[v]
13    Q.buildHeap(G.vertices())
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21          d[v] = cost(v, m)
22          p[v] = m
23
```

line 16 & 19 → m x n times inside for loop

If dense equal to or better than in all operations

	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$ 10%
Unsorted Array	$O(n^2)$ add super agree	$O(n^2)$ 70%

Prim's Algorithm

Sparse Graph: $m \sim n$

Adj List Heap best

Dense Graph: $m \sim n^2$

Unsorted Array best

```
 6  PrimMST(G, s):
 7    foreach (Vertex v : G.vertices()):
 8      d[v] = +inf
 9      p[v] = NULL
10    d[s] = 0
11
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21          d[v] = cost(v, m)
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23
```

	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$
Unsorted Array	$O(n^2)$	$O(n^2)$

MST Algorithm Runtime:



Kruskal's Algorithm:

O(n + m log (n))

$\log(m) \simeq \log(n)$

Prim's Algorithm:

O(n log(n) + m log (n))

Sparse Graph: m ~ n

$n + n \log n$

$n \log n + n \log n$

$O(n \log n)$

Dense Graph: $m \sim n^2$

$n^2 \log n$

$n^2 \log n$

Same here too * → Prims do unsorted array in $O(n^2)$

Suppose I have a new heap:

1984 / 1987 Michael Fredman Robert Tarjan

	Binary Heap	Fibonacci Heap
Remove Min	O(lg(n))	O(lg(n))
Decrease Key	O(lg(n))	O(1)*

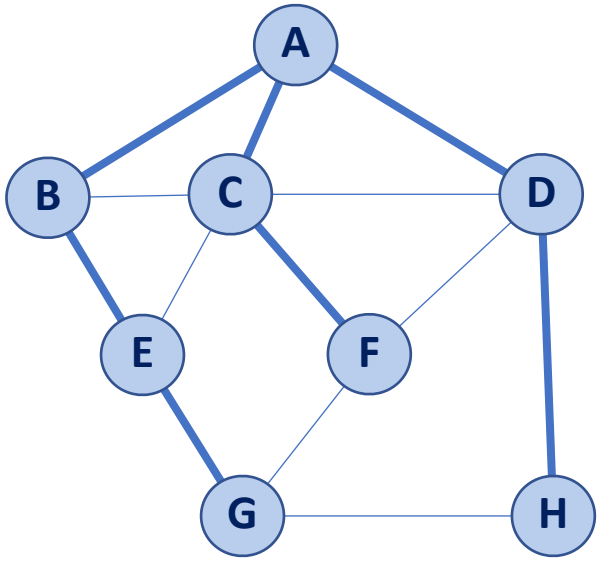
What's Prim's updated running time?

$O(n \log n + m)$

```
    PrimMST(G, s):
 6    foreach (Vertex v : G.vertices()):
 7      d[v] = +inf
 8      p[v] = NULL
 9    d[s] = 0
10
11    PriorityQueue Q // min distance, defined by d[v]
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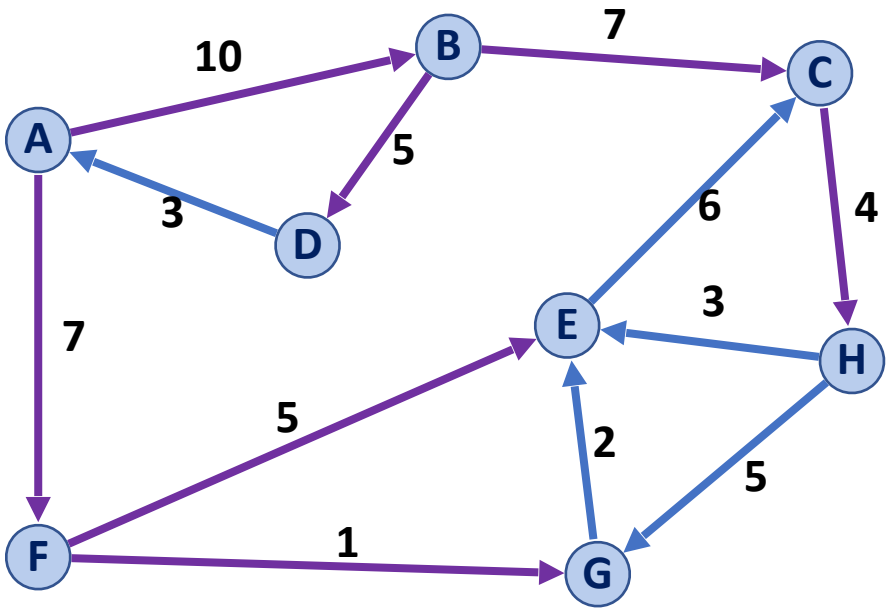
$O(\log n)$ → $O(1)^*$

Shortest Path





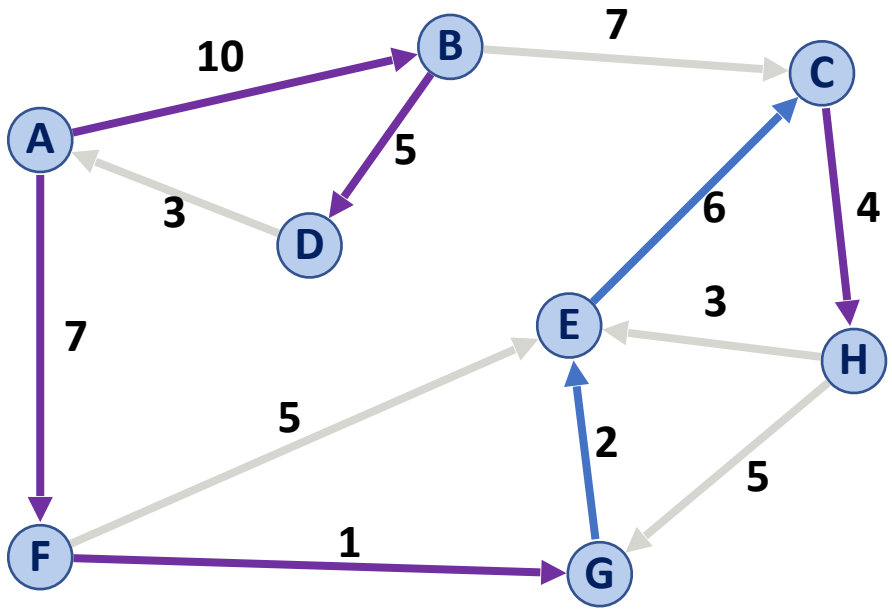
Dijkstra's Algorithm (SSSP)




```
    DijkstraSSSP(G, s):
 6    foreach (Vertex v : G.vertices()):
 7      d[v] = +inf
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 9    d[s] = 0
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11    PriorityQueue Q // min distance, defined by d[v]
12    Q.buildHeap(G.vertices())
13    Graph T           // "labeled set"
14
15    repeat n times:
16      Vertex u = Q.removeMin()
17      T.add(u)
18      foreach (Vertex v : neighbors of u not in T):
19        if _____________< d[v]:
20          d[v] = _____________
21          p[v] = u
```

A	B	C	D	E	F	G	H
--							
0							

Dijkstra's Algorithm (SSSP)




```
    DijkstraSSSP(G, s):
 6    foreach (Vertex v : G.vertices()):
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19        if cost(u, v) + d[u] < d[v]:
20          d[v] = cost(u, v) + d[u]
21          p[v] = u
```

A	B	C	D	E	F	G	H
--	A	E	B	G	A	F	C
0	10	16	15	10	7	8	20

Dijkstra's Algorithm (SSSP)

What is the running time of Dijkstra's Algorithm?

```
    DijkstraSSSP(G, s):
 6    foreach (Vertex v : G):
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21          p[v] = m
22
23    return T
```

Dijkstra's Algorithm (SSSP)

What is the running time of Dijkstra's Algorithm?

O(m + n log n)

The same as Prim's!

(Times here are minheap)

6-9: O(n)

11-12: O(n)

15: repeat below n x

16-22: O(log n)

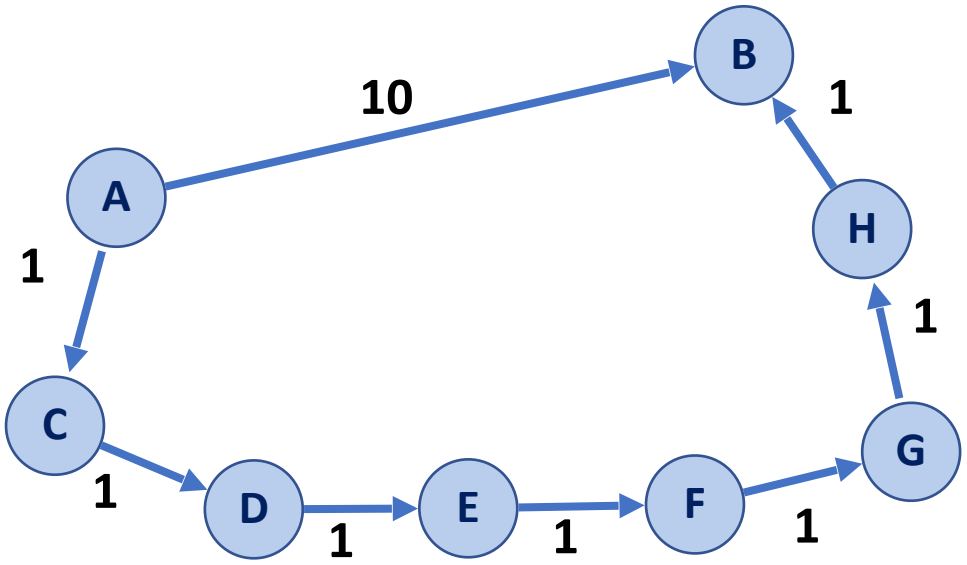
[w/ Fib Heap O(1) updates]

```
    DijkstraSSSP(G, s):
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17      T.add(u)
18      foreach (Vertex v : neighbors of u not in T):
19        if cost(u, v) + d[u] < d[v]:
20          d[v] = cost(u, v) + d[u]
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```

Dijkstra's Algorithm (SSSP)

Claim: Dijkstras will always visit a node through its optimal shortest path.

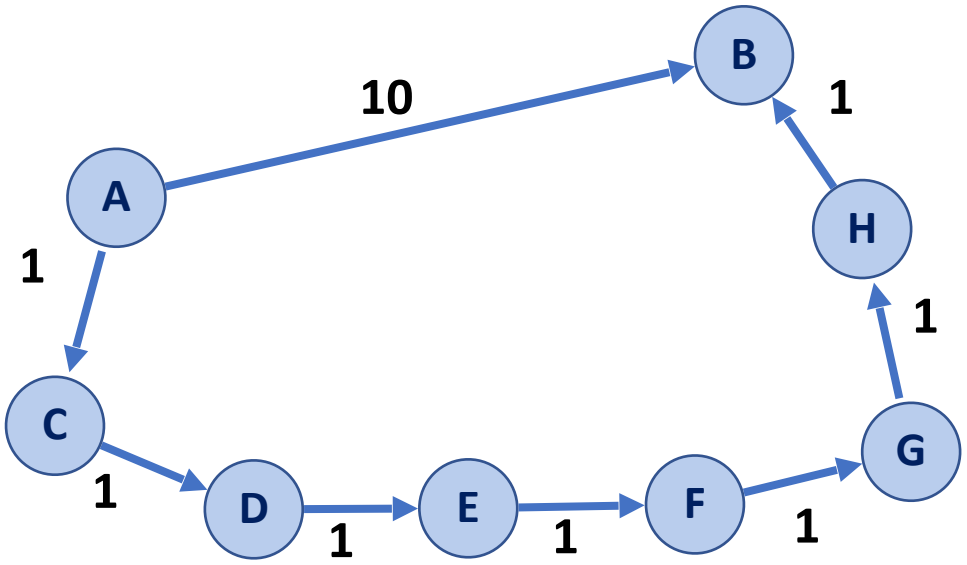
When we will visit B in the following graph?



Dijkstra's Algorithm (SSSP)

Claim: Dijkstras will always visit a node through its optimal shortest path.

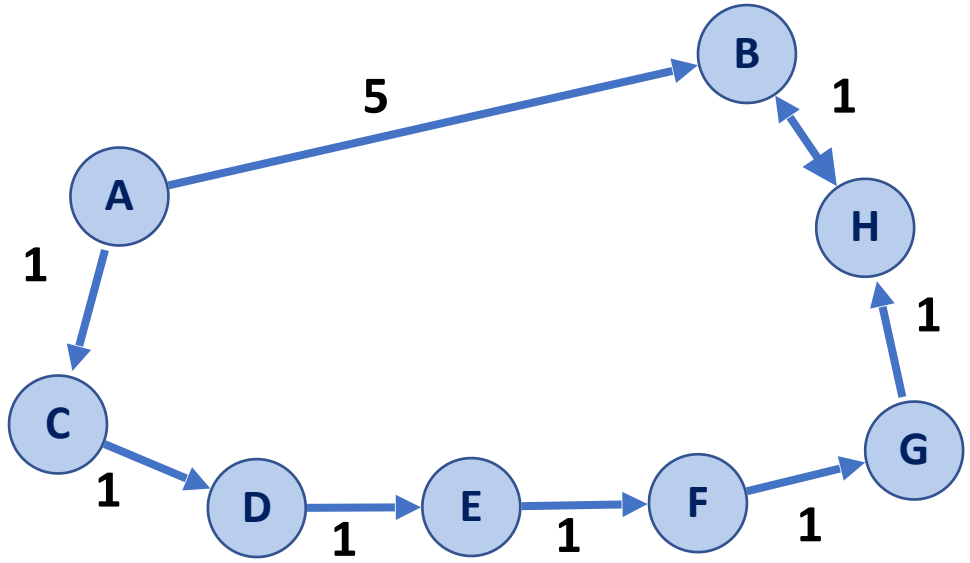
When we will visit H in the following graph?



Dijkstra's Algorithm (SSSP)

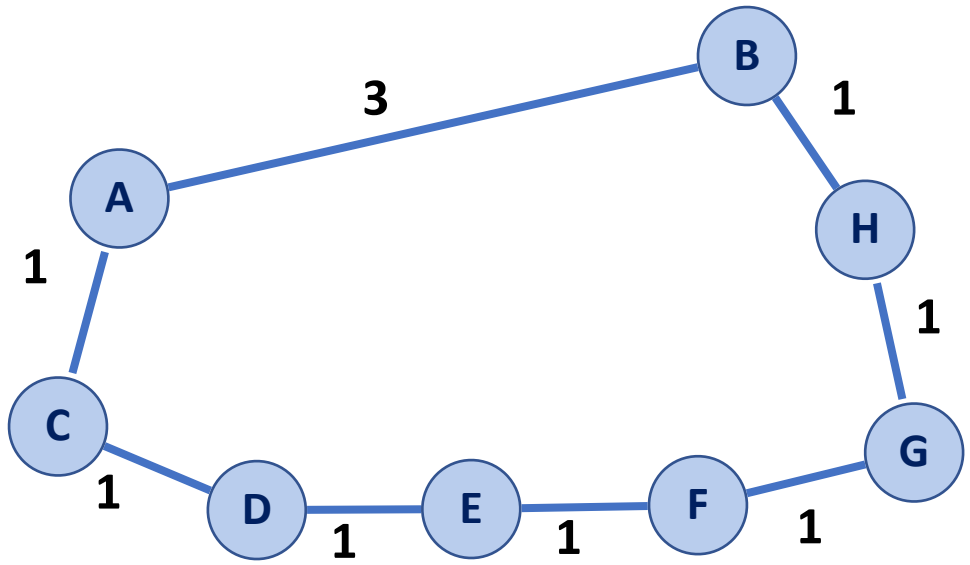
Claim: Dijkstras will always visit a node through its optimal shortest path.

When we will visit B in the following graph?



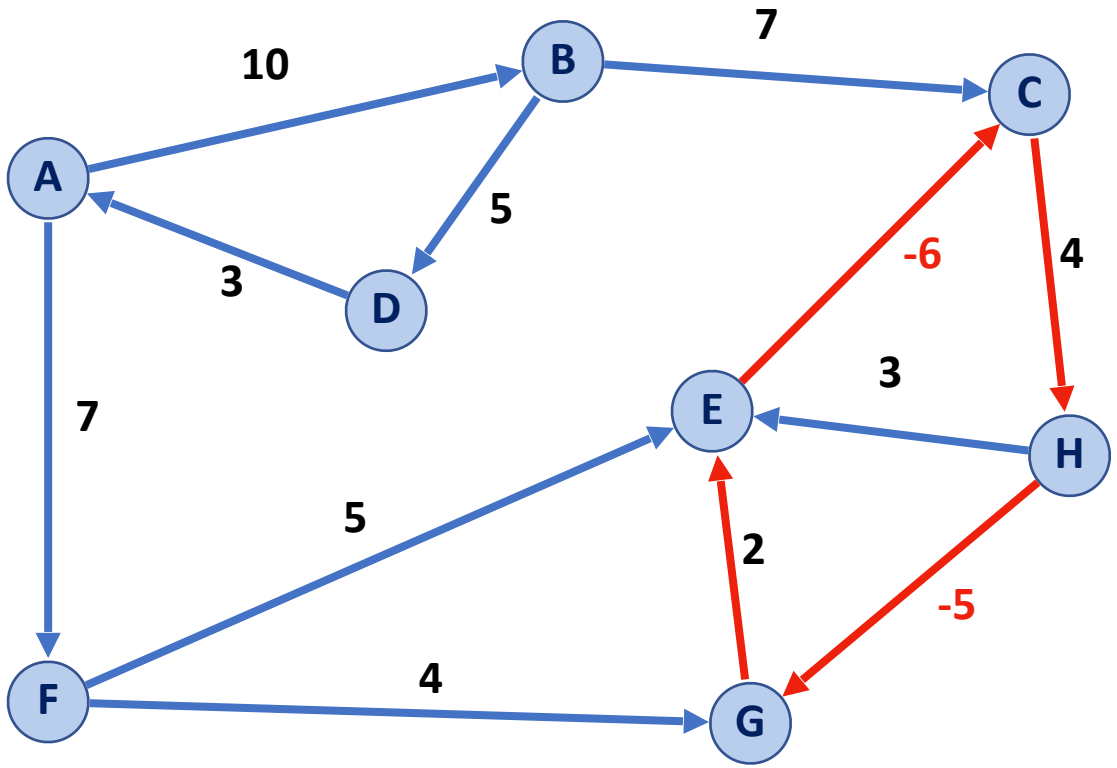
Dijkstra's Algorithm (SSSP)

How does Dijkstra's algorithm handle undirected graphs?



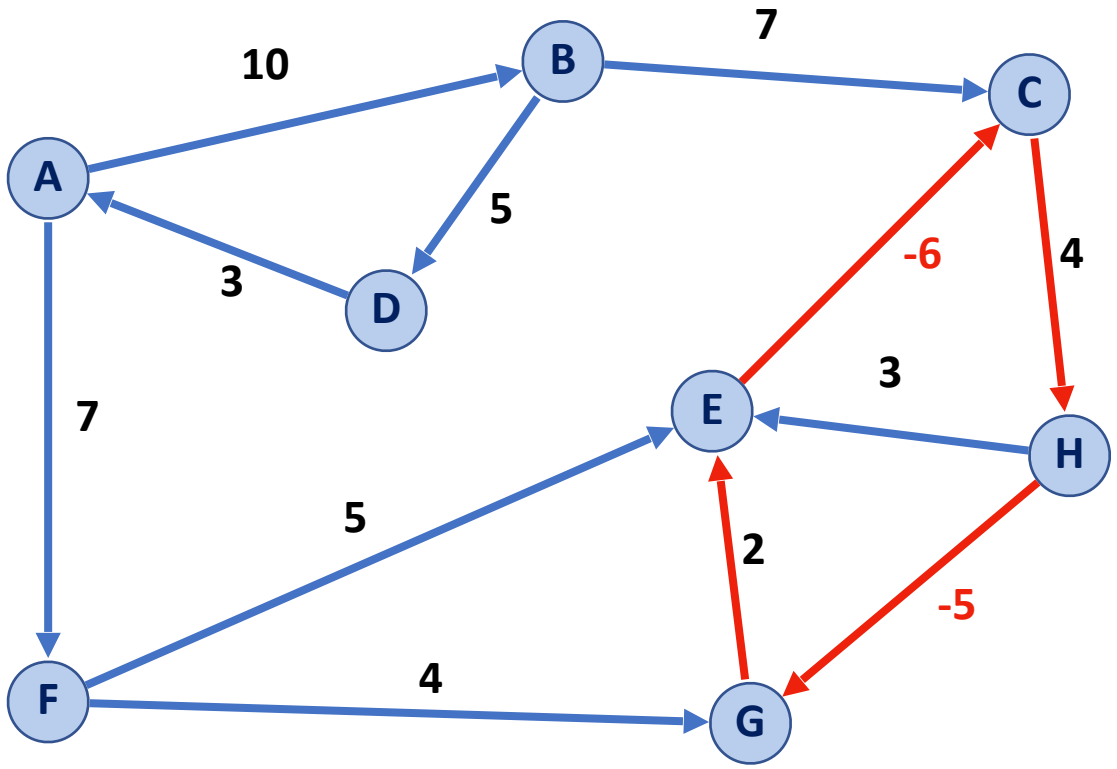
Dijkstra's Algorithm (SSSP)

How does Dijkstras handle a negative weight cycle?



Dijkstra's Algorithm (SSSP)

How does Dijkstras handle a negative weight cycle?



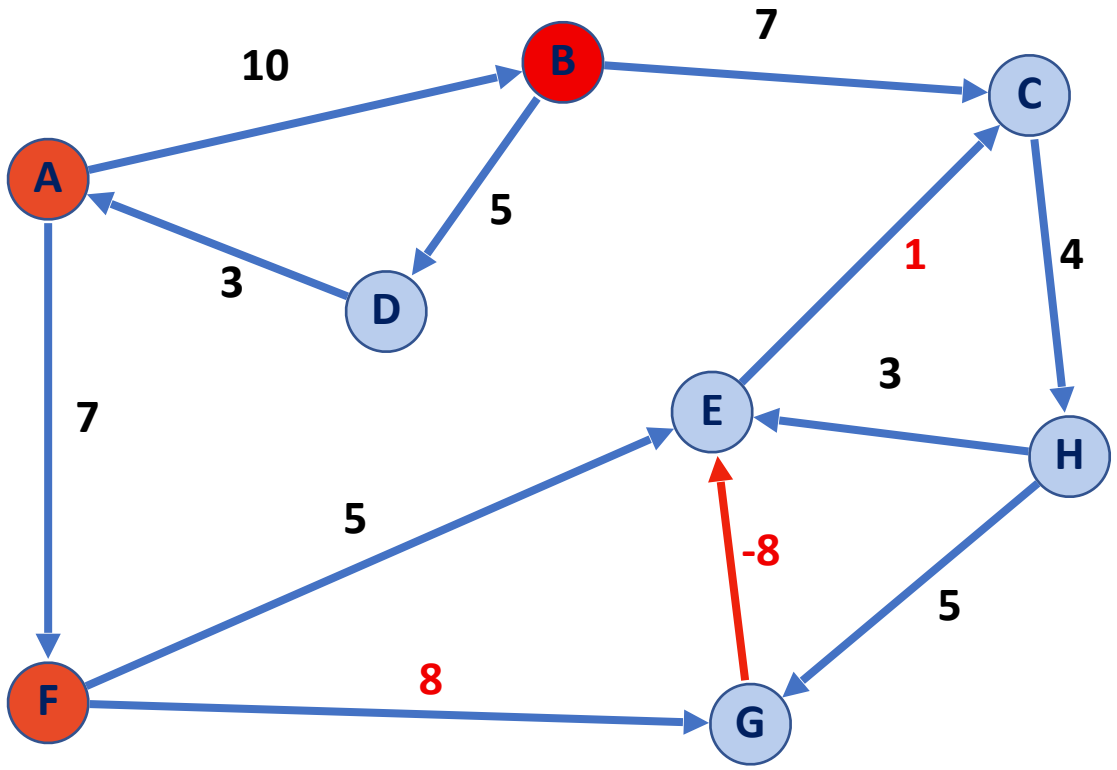
Shortest Path (A → E): A → F → E → (C → H → G → E)*

Length: 12

Length: -5 (repeatable)

Dijkstra's Algorithm (SSSP)

How does Dijkstras handle a negative weight edge without a cycle?



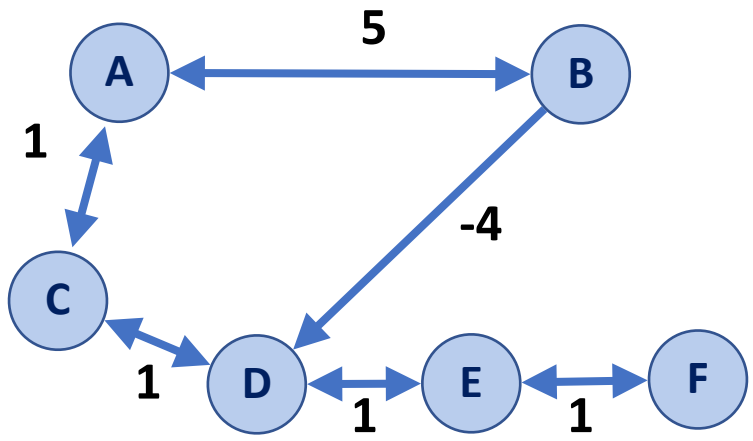
A	B	C	D	E	F	G	H
--	A	B	B	F	A	F	--
0	10	17	15	12	7	15	∞

Dijkstra's Algorithm (SSSP)

We assume that item pulled out of priority queue is **the next smallest item**

Negative weights break this assumption!

A	B	C	D	E	F
--					
0					



Dijkstra's Algorithm (SSSP)

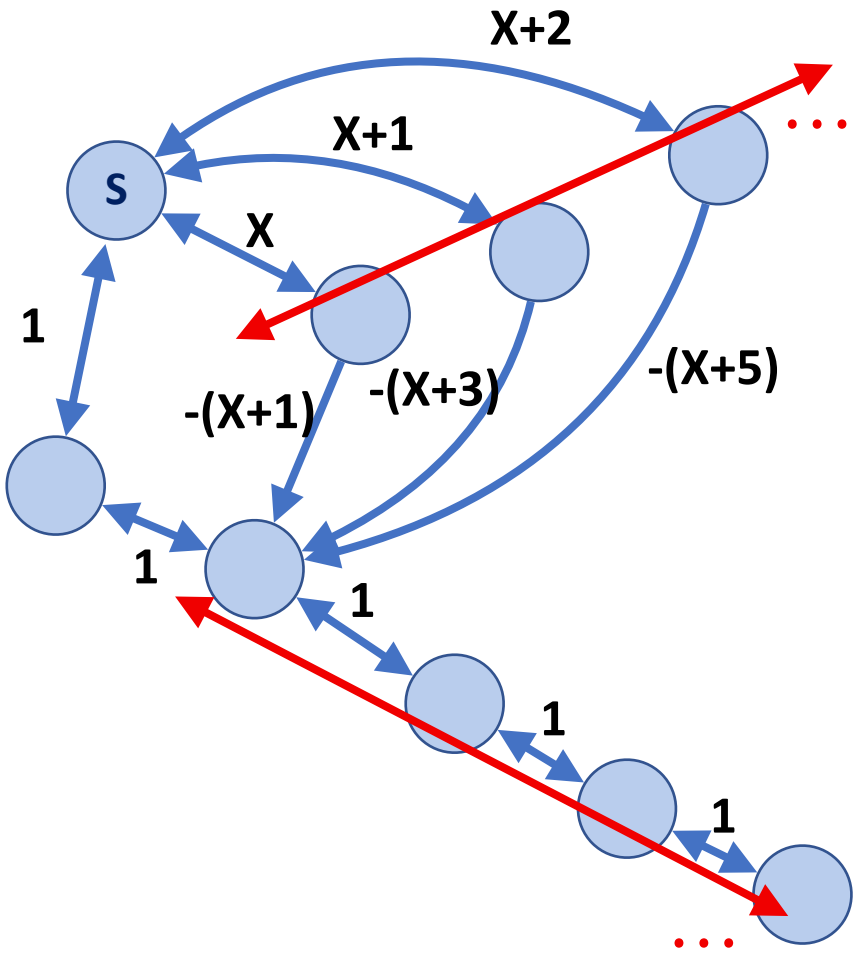
Recalculating all distances is possible, but algorithm runtime is very bad!

```
    DijkstraSSSP(G, s):
 6    foreach (Vertex v : G):
 7      d[v] = +inf
 8      p[v] = NULL
 9    d[s] = 0
10
11    PriorityQueue Q // min distance, defined by d[v]
12    Q.buildHeap(G.vertices())
13    Graph T          // "labeled set"
14
15    repeat until Q.empty():
16      Vertex u = Q.removeMin()
17      T.add(u)
18      foreach (Vertex v : neighbors of u not in T):
19        if cost(u, v) + d[u] < d[v]:
20          d[v] = cost(u, v) + d[u]
21          p[v] = m
22          if v not in Q:
23             Q.push(v)
24    return T
```

Dijkstra's Algorithm (SSSP)

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Worst case: ~n/2 nodes each updating ~n/2 nodes distances




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```

Dijkstra's Algorithm (SSSP)



Dijkstras Algorithm works only on non-negative weights

Optimal implementation:

Fibonacci Heap

If dense, unsorted list ties

Optimal runtime:

Sparse: O(m + n log n)

Dense: $O(n^2)$

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10
11    PriorityQueue Q // min distance, defined by d[v]
12    Q.buildHeap(G.vertices())
13    Graph T          // "labeled set"
14
15    repeat n times:
16      Vertex u = Q.removeMin()
17      T.add(u)
18      foreach (Vertex v : neighbors of u not in T):
19        if cost(u, v) + d[u] < d[v]:
20          d[v] = cost(u, v) + d[u]
21          p[v] = m
22
23    return T
```

Landmark Path Problem

What if I wanted to get the shortest path from A to G but stopping at L along the way?

