

Data Structures

AVL Analysis

CS 225

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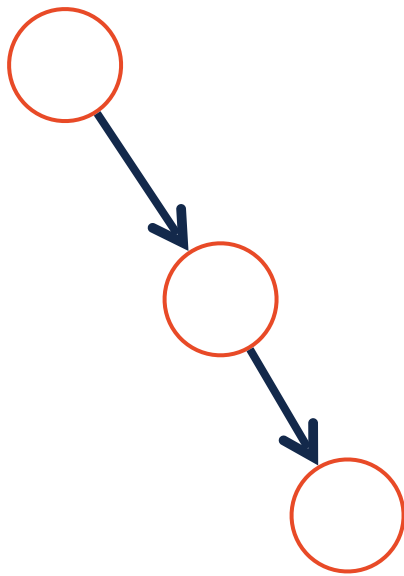
Learning Objectives

Review AVL trees

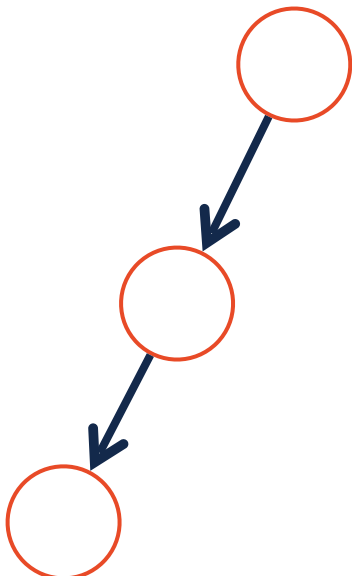
Prove that the AVL Tree speeds up all operations

AVL Rotations

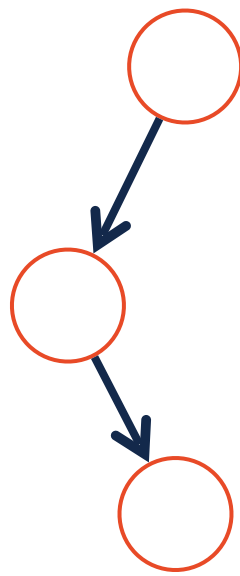
Left



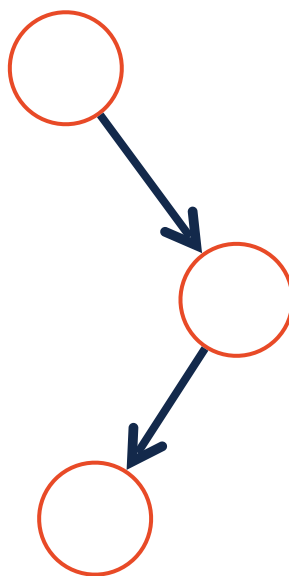
Right



LeftRight



RightLeft



Root Balance: 2

-2

-2

2

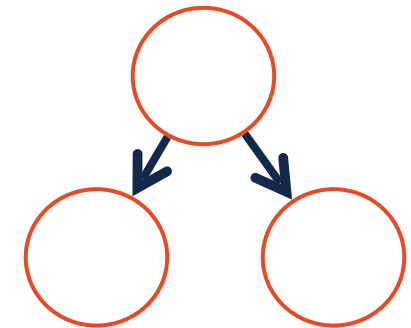
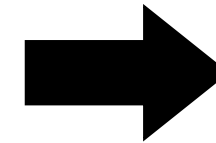
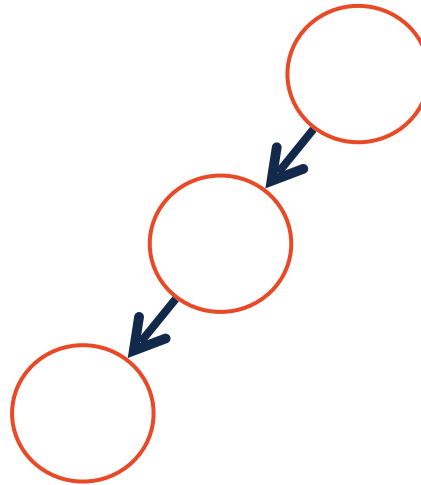
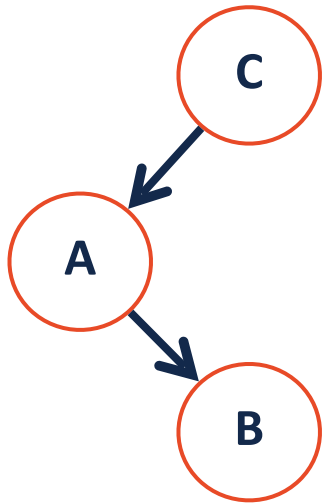
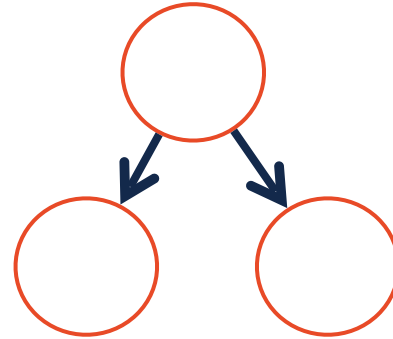
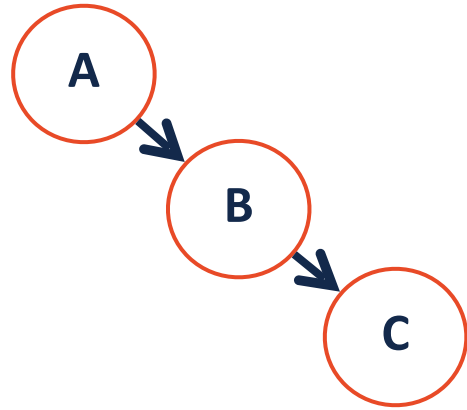
Child Balance: 1

-1

1

-1

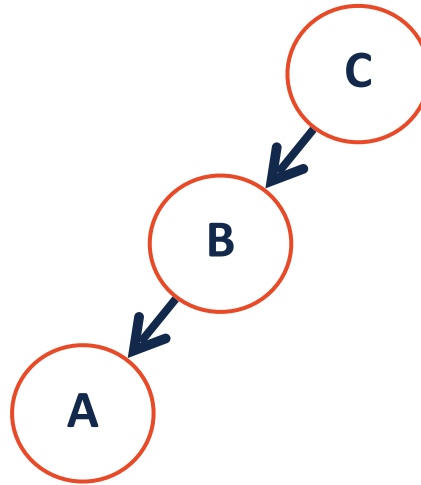
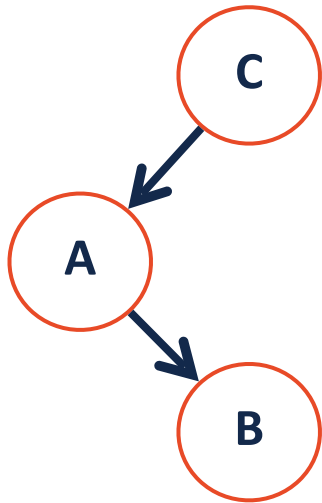
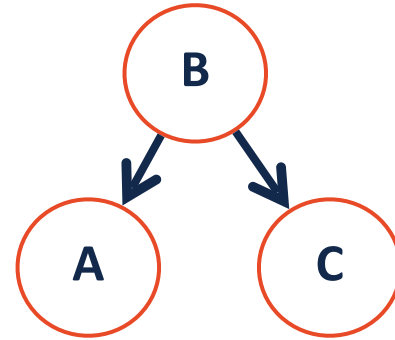
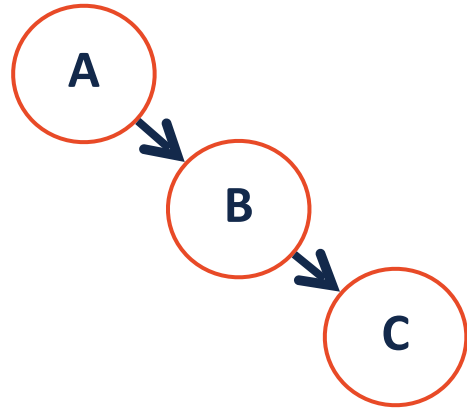
AVL Tree Rotations



All rotations are $O(1)$

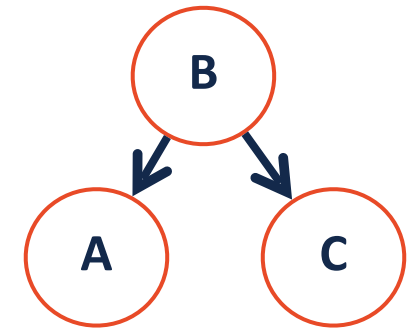
All rotations reduce subtree height by one

AVL Tree Rotations



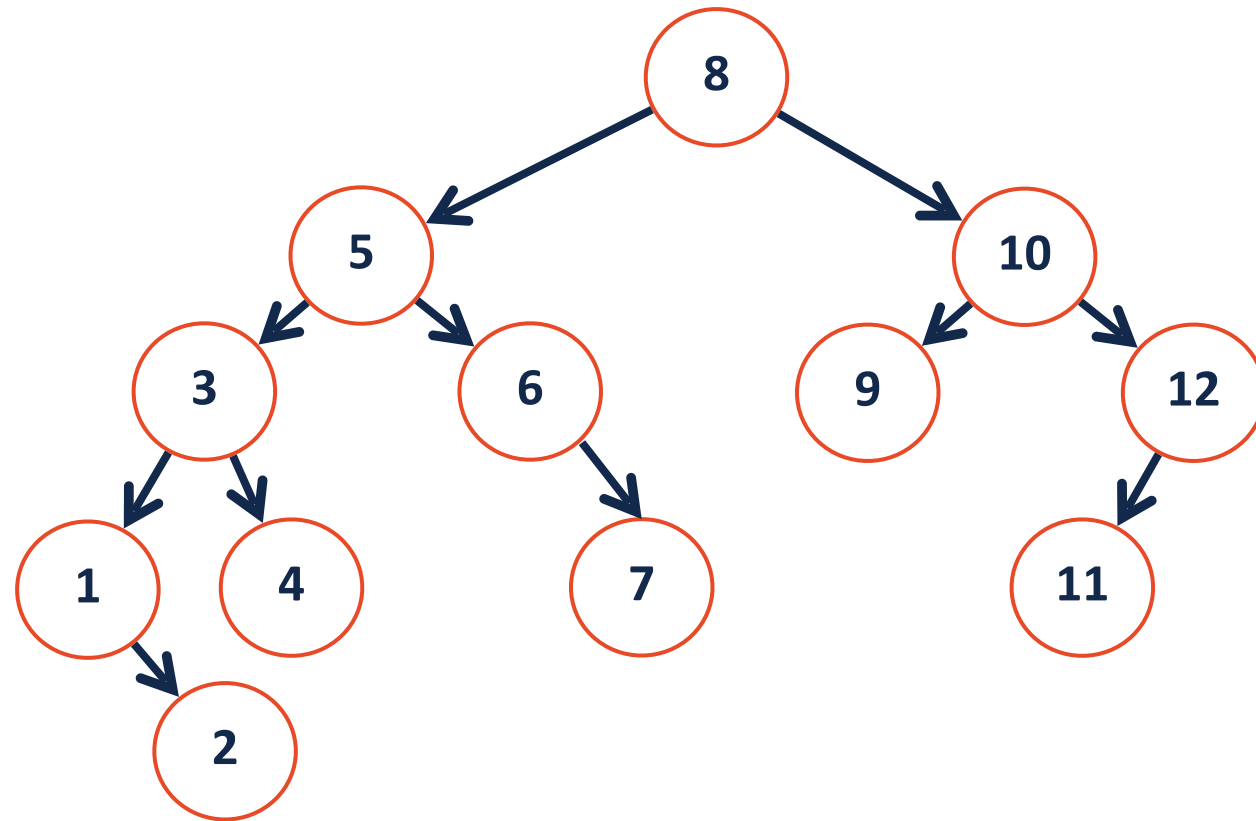
All rotations are $O(1)$

All rotations reduce subtree height by one



AVL Insertion

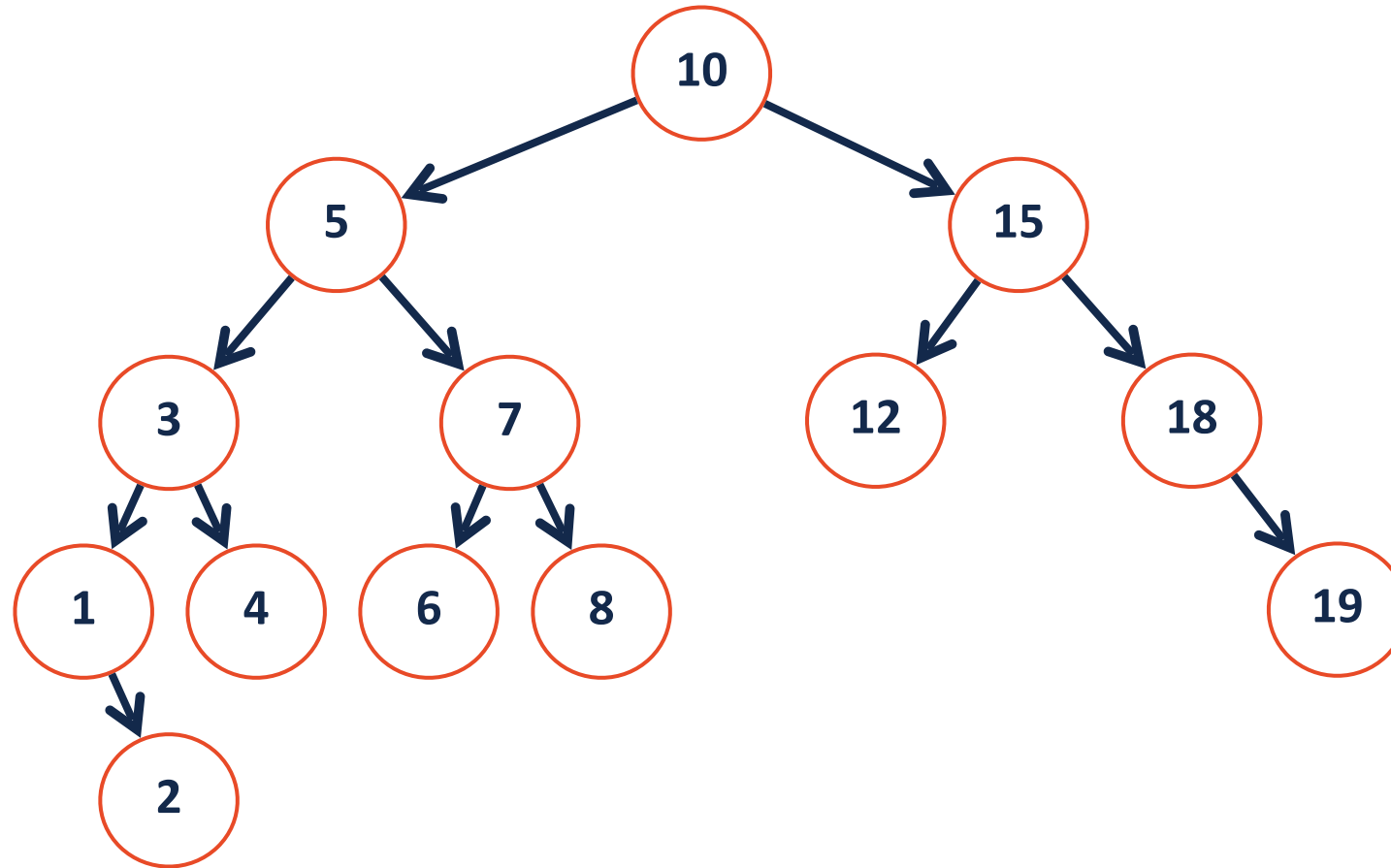
Given an AVL is balanced, insert can insert **at most** one imbalance



AVL Remove

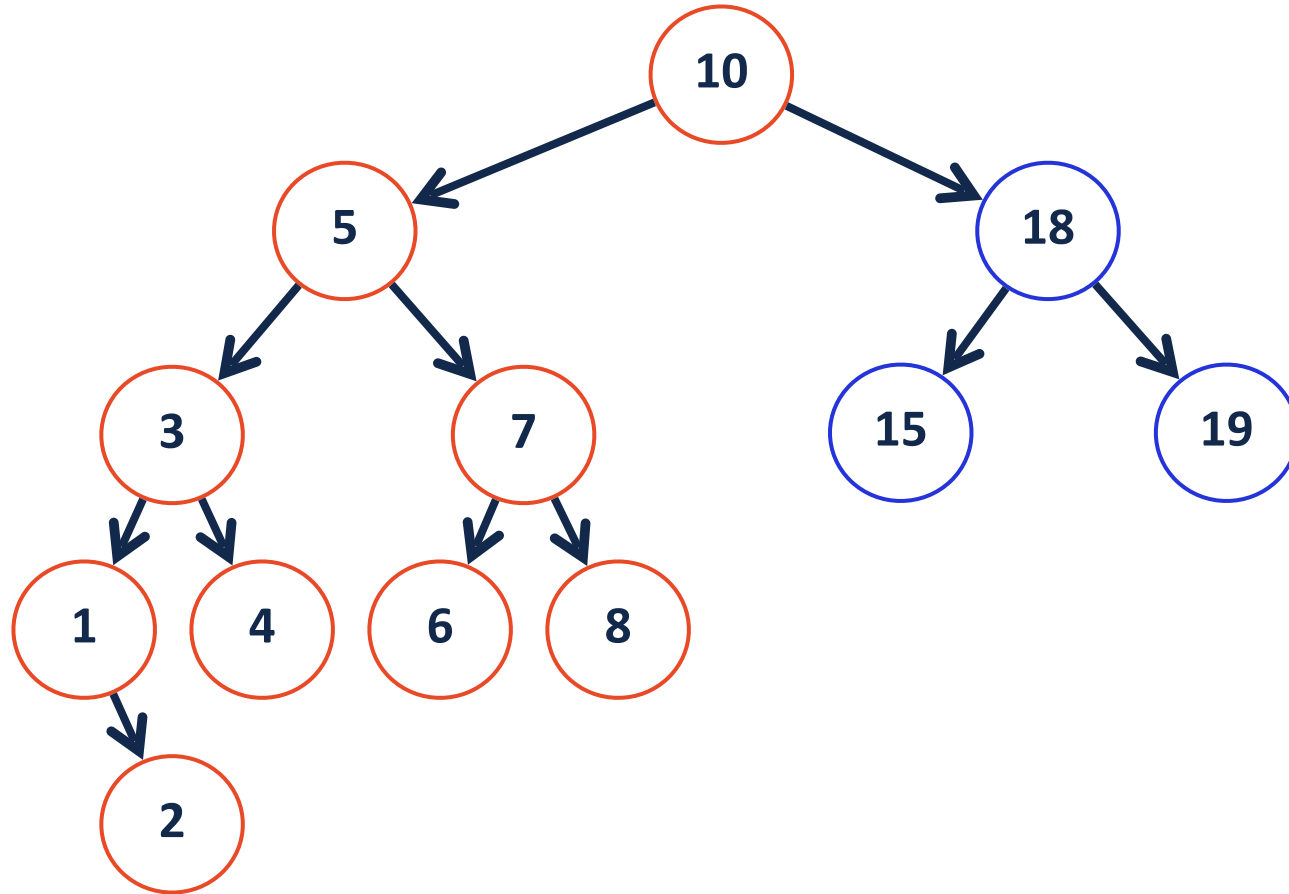
`_remove(12)`

Remove can cause an imbalance at every level



AVL Remove

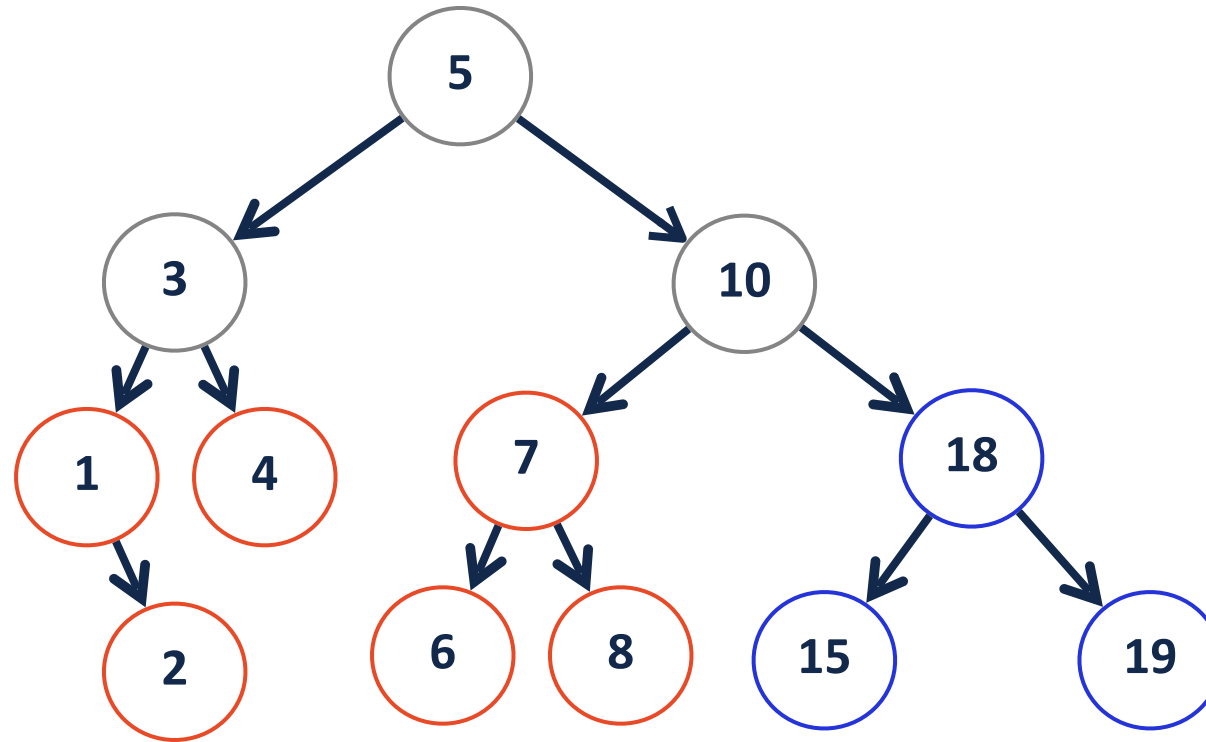
Remove can cause an imbalance at every level



AVL Remove

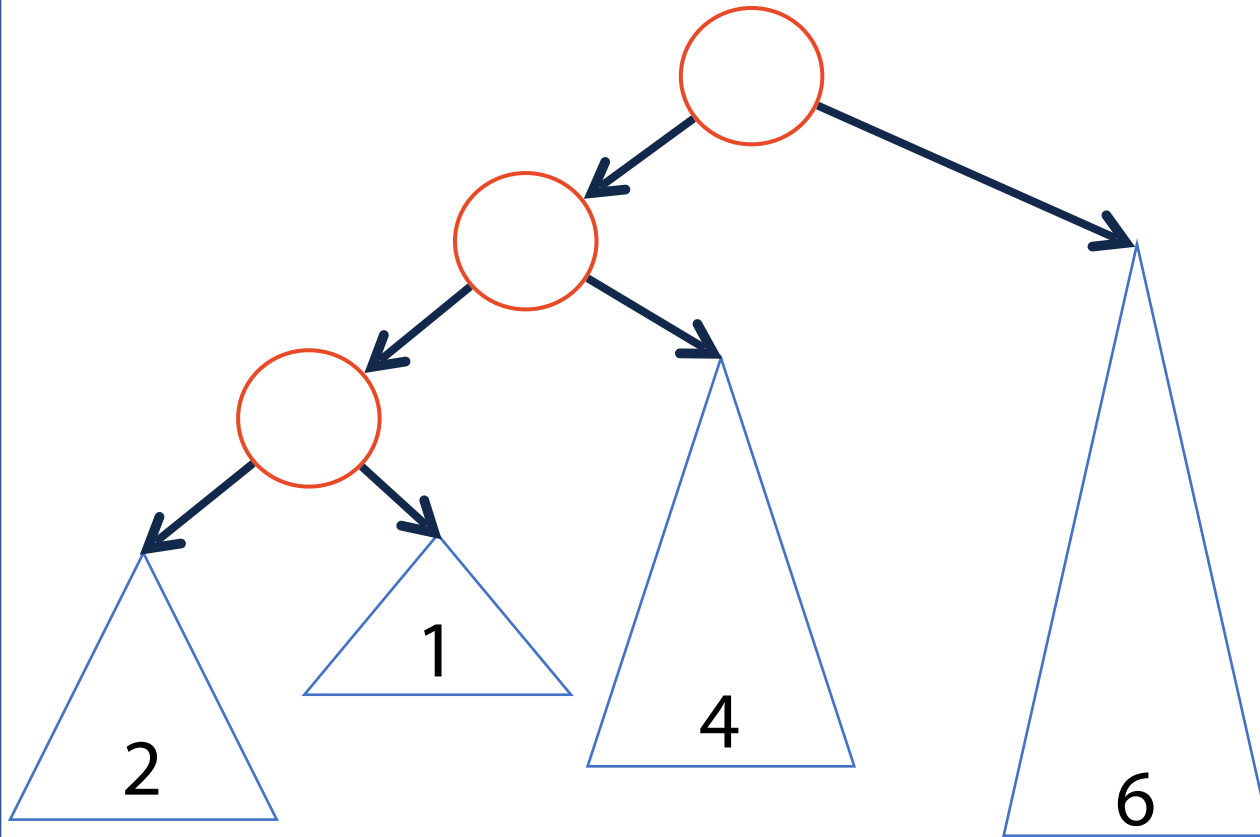
`_remove(12)`

Remove can cause an imbalance at every level



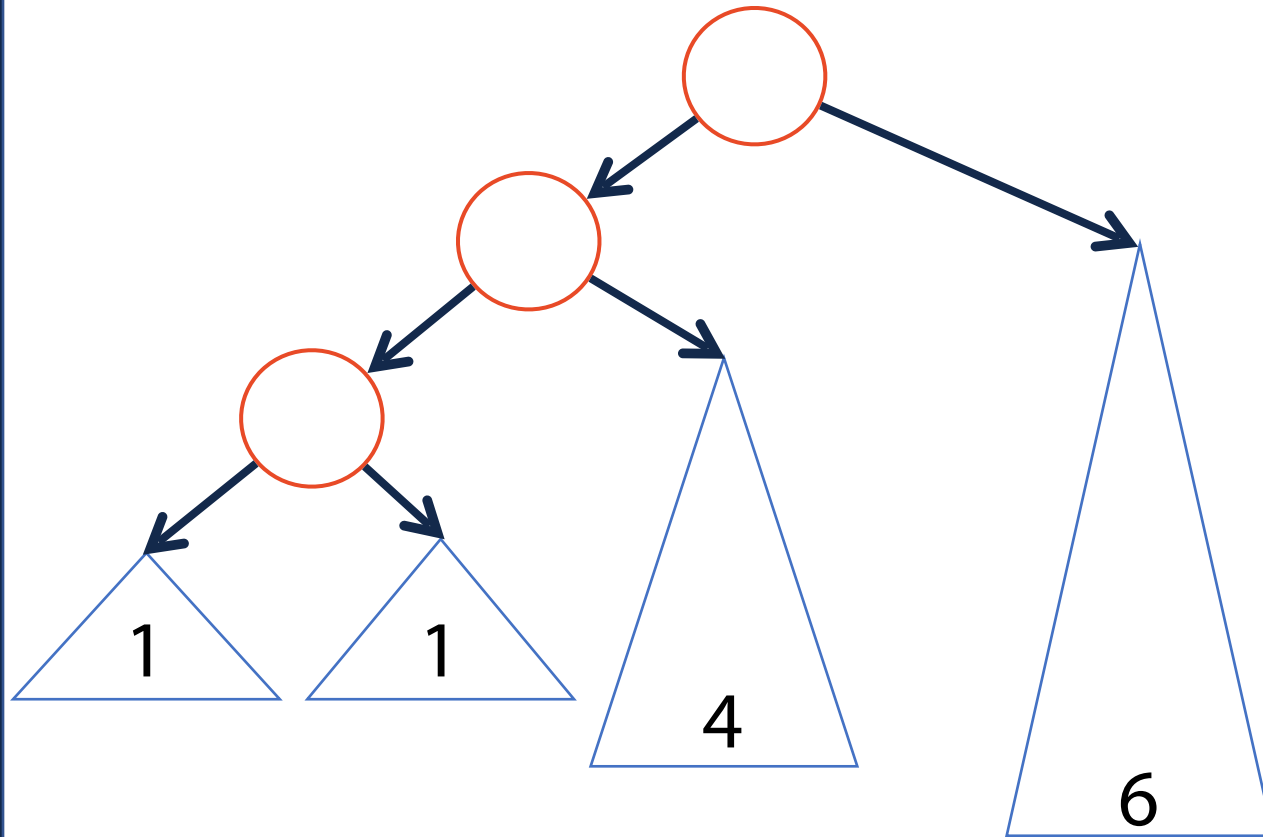
AVL Remove

Remove can cause an imbalance at every level



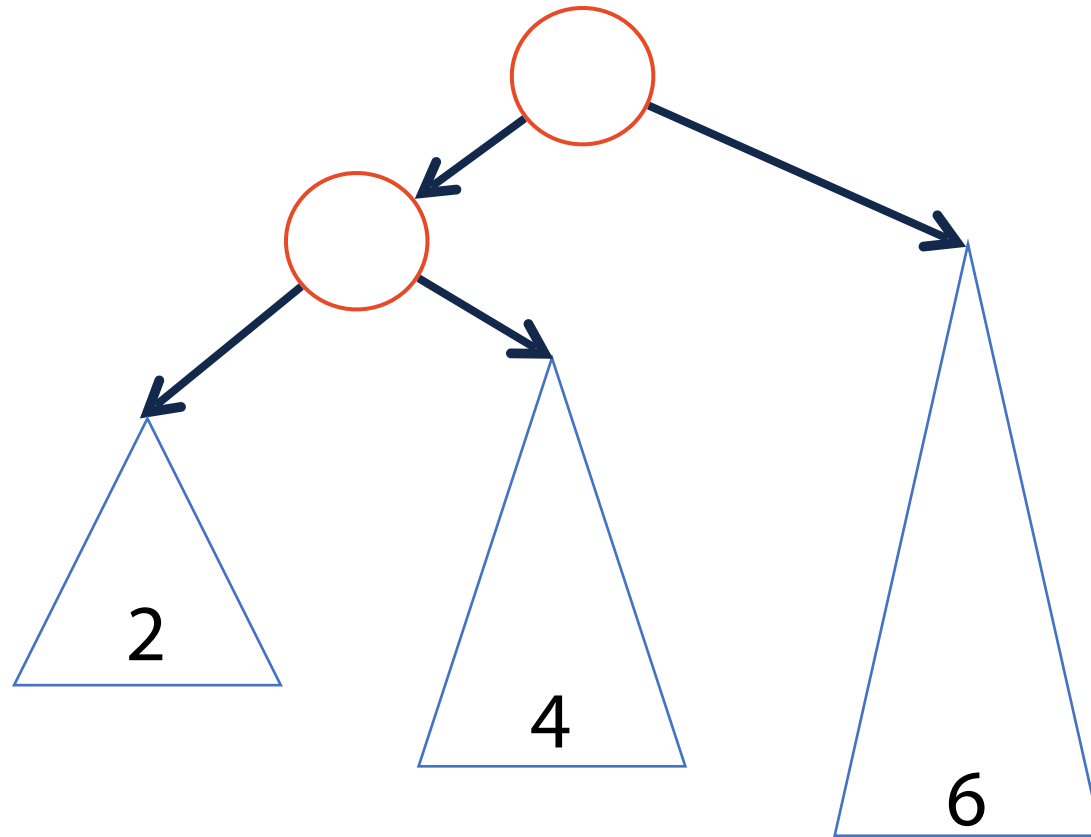
AVL Remove

Remove can cause an imbalance at every level



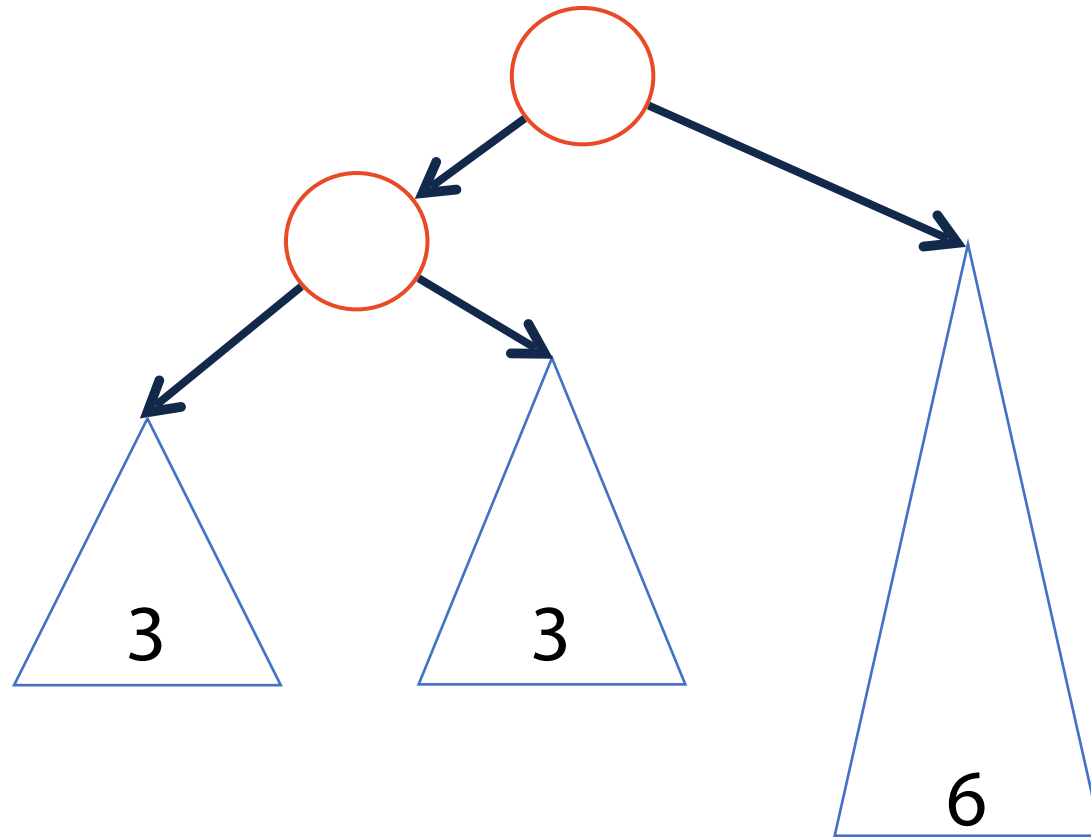
AVL Remove

Remove can cause an imbalance at every level



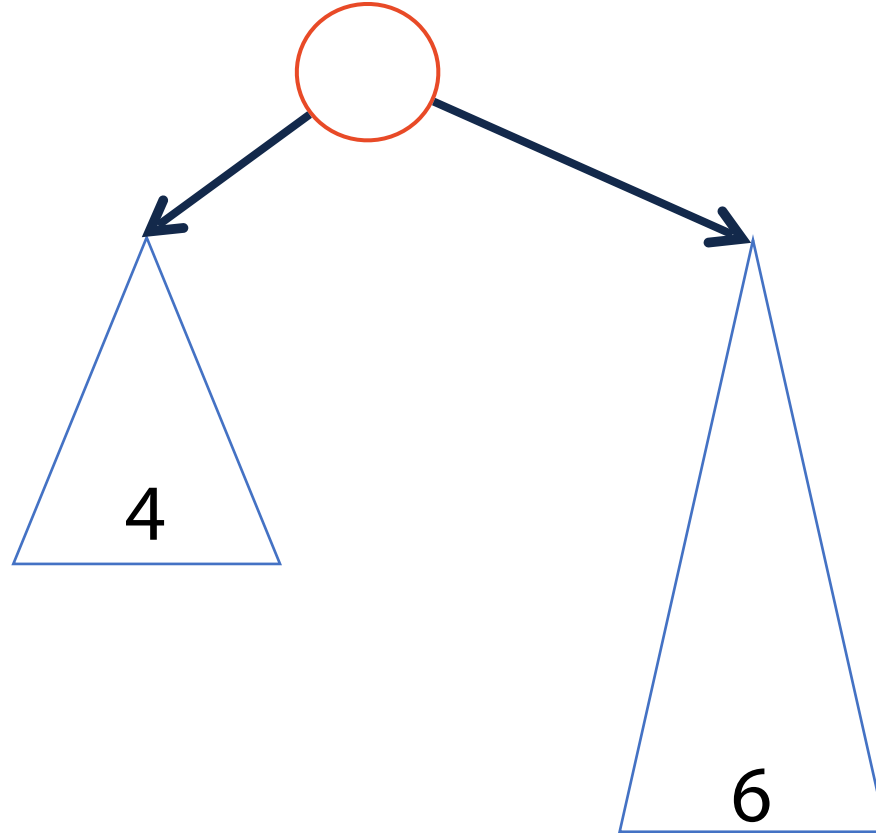
AVL Remove

Remove can cause an imbalance at every level



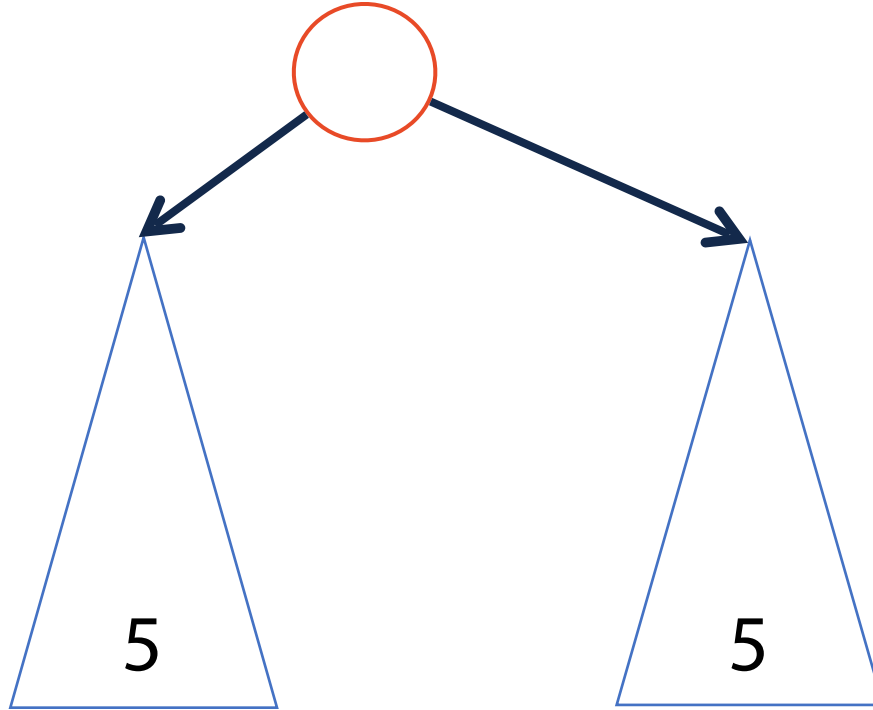
AVL Remove

Remove can cause an imbalance at every level



AVL Remove

Remove can cause an imbalance at every level



AVL Tree Analysis



For an AVL tree of height h :

Find runs in: _____.

Insert runs in: _____.

Remove runs in: _____.

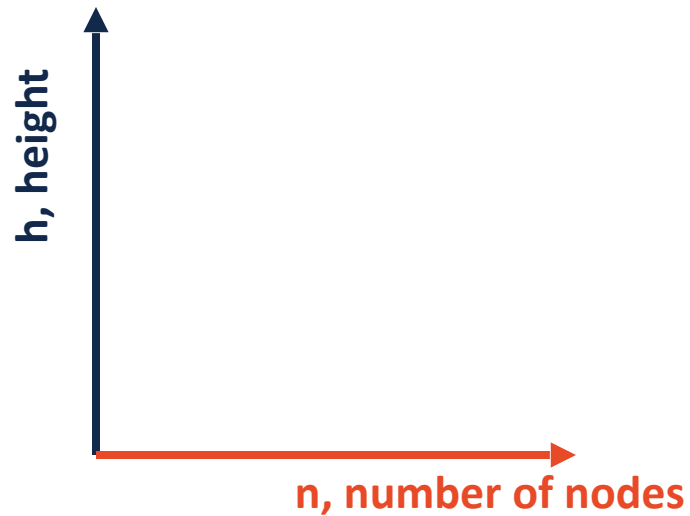
Claim: The height of the AVL tree with n nodes is: _____.

AVL Tree Analysis

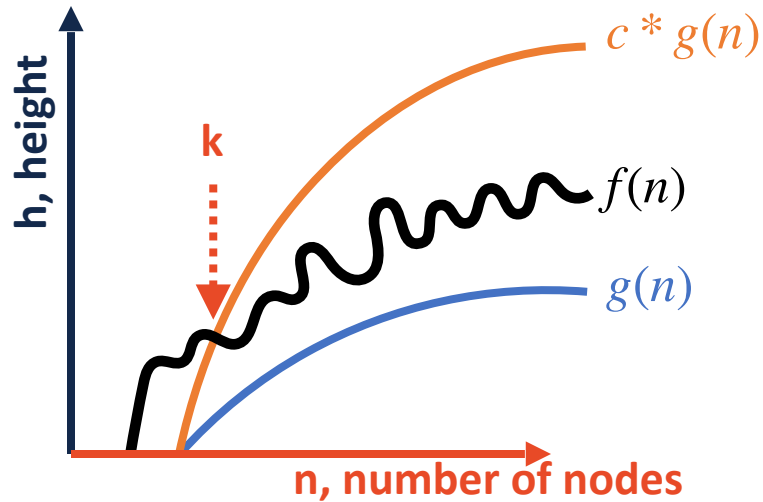
Definition of big-O:

$$f(n) \text{ is } O(g(n)) \text{ iff } \exists c, k \text{ s.t. } f(n) \leq cg(n) \quad \forall n > k$$

...or, with pictures:

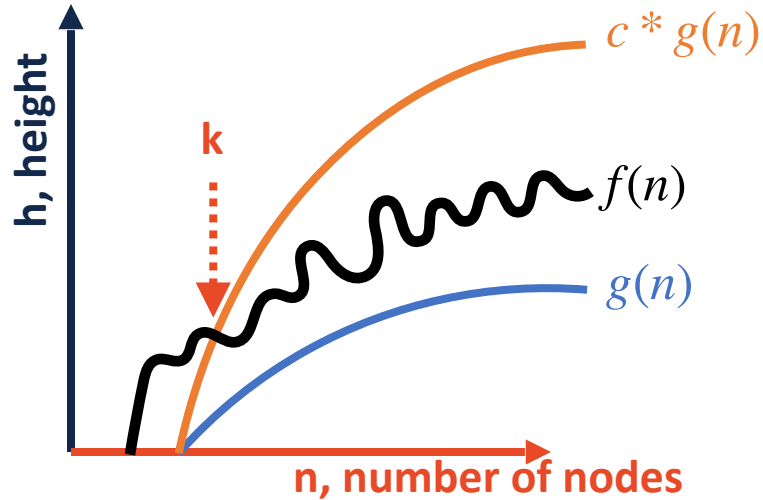


AVL Tree Analysis

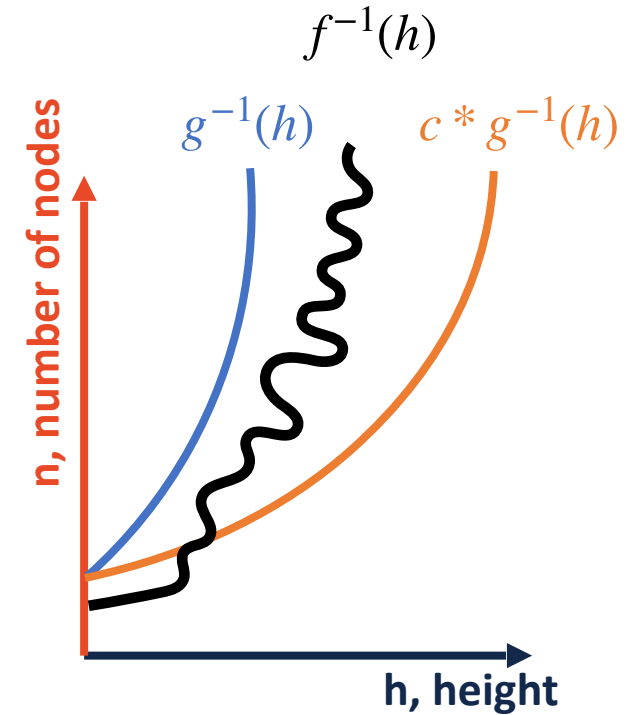


The height of the tree, $f(n)$, will always be less than $c \times g(n)$ for all values where $n > k$.

AVL Tree Analysis



$f(n)$ = "Tree height given nodes"



$f^{-1}(h)$ = "Nodes in tree given height"

The number of nodes in the tree, $f^{-1}(h)$, will always be greater than $c \times g^{-1}(h)$ for all values where $n > k$.

Plan of Action

Since our goal is to find the lower bound on n given h , we can begin by defining a function given h which describes the smallest number of nodes in an AVL tree of height h :

$N(h)$ = minimum number of nodes in an AVL tree of height h

Simplify the Recurrence

$$N(h) = 1 + N(h - 1) + N(h - 2)$$

Simplify the Recurrence

$$N(h) = 1 + N(h - 1) + N(h - 2)$$

$$N(h) > N(h - 1) + N(h - 2)$$

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$$N(h) = 1 + N(h - 1) + N(h - 2)$$

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$$N(h) > 2N(h - 2)$$

Simplify the Recurrence

$$N(h) = 1 + N(h - 1) + N(h - 2)$$

$$N(h) > N(h - 1) + N(h - 2)$$

$$N(h) > 2N(h - 2)$$

1) Know characteristic equation? Get answer immediately!

Simplify the Recurrence

$$N(h) = 1 + N(h - 1) + N(h - 2)$$

$$N(h) > N(h - 1) + N(h - 2)$$

$$N(h) > 2N(h - 2)$$

$$2) \text{ Unroll: } N(h) > 2N(h - 2) = 2 \left(2(N(h - 4)) \right) = 2^k \left(N(h - 2k) \right)$$

Simplify the Recurrence

$$N(h) = 1 + N(h - 1) + N(h - 2)$$

$$N(h) > N(h - 1) + N(h - 2)$$

$$N(h) > 2N(h - 2)$$

$$2) \text{ Unroll: } N(h) > 2N(h - 2) = 2(2N(h - 4)) = 2^k(N(h - 2k))$$

When $h - 2k = 0$, $k = h/2$. Thus $N(h) > 2^{h/2}$

Simplify the Recurrence

$$N(h) = 1 + N(h - 1) + N(h - 2)$$

$$N(h) > N(h - 1) + N(h - 2)$$

$$N(h) > 2N(h - 2)$$

3) Intuit approximate shape from recursion

Simplify the Recurrence

$$N(h) = 1 + N(h - 1) + N(h - 2)$$

$$N(h) > N(h - 1) + N(h - 2)$$

$$N(h) > 2N(h - 2)$$

By whatever strategy you like: $N(h) > 2^{h/2}$

State a Theorem

Theorem: An AVL tree of height h has at least $N(h) > 2^{h/2}$.

Proof by Induction:

I. Consider an AVL tree and let h denote its height.

II. Base Case: _____

An AVL tree of height _____ has at least _____ nodes.

Prove a Theorem

III. Base Case: _____

An AVL tree of height _____ has at least _____ nodes.

Prove a Theorem

IV. Induction Step: Assume for all heights $i < h$, $N(i) \geq 2^{i/2}$.

Prove that $N(h) \geq 2^{h/2}$

Prove a Theorem

IV. Induction Step: Assume for all heights $i < h$, $N(i) \geq 2^{i/2}$.

Prove that $N(h) \geq 2^{h/2}$

$$N(h) = 1 + N(h-1) + N(h-2)$$

$$N(h) > 2N(h-2)$$

$$N(h) > 2 * 2^{(h-2)/2}$$

$$N(h) > 2 * 2^{h/2-1}$$

$$N(h) > 2^{h/2}$$

Prove a Theorem

V. Using a proof by induction, we have shown that:

Prove a Theorem

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$N(h) \geq 2^{h/2}$, where $N(h)$ is the **min # of nodes of a tree of height h**

But we need to know n , the **# of nodes in any tree of height h**



Prove a Theorem

V. Using a proof by induction, we have shown that:

$N(h) \geq 2^{h/2}$, where $N(h)$ is the **min # of nodes of a tree of height h**

But we need to know n , the **# of nodes in any tree of height h**

$$n \geq N(h)$$

$$\log(n) \geq \frac{h}{2}$$

$$h \leq 2 \log(n)$$

AVL Runtime Proof

An upper-bound on the height of an AVL tree is **$O(\lg(n))$** :

$N(h)$:= Minimum # of nodes in an AVL tree of height h

$$N(h) = 1 + N(h-1) + N(h-2)$$

$$> 1 + 2^{(h-1)/2} + 2^{(h-2)/2}$$

$$> 2 \times 2^{(h-2)/2} = 2^{(h-2)/2+1} = 2^{h/2}$$

Theorem #1:

Every AVL tree of height h has at least $2^{h/2}$ nodes.

AVL Runtime Proof

An upper-bound on the height of an AVL tree is $O(\lg(n))$:

$$\text{\# of nodes } (n) \geq N(h) > 2^{h/2}$$

$$n > 2^{h/2}$$

$$\lg(n) > h/2$$

$$2 \times \lg(n) > h$$

$$h < 2 \times \lg(n) \quad , \text{ for } h \geq 1$$

Proved: The maximum number of nodes in an AVL tree of height h is less than $2 \times \lg(n)$.



Summary of Balanced BST

Pros:

Cons:

Summary of Balanced BST

AVL Trees

- Max height: $1.44 * \lg(n)$
- Rotations:

Summary of Balanced BST

AVL Trees

- Max height: $1.44 * \lg(n)$

- Rotations:

 - Zero rotations on find

 - One rotation on insert

 - $O(h) == O(\lg(n))$ rotations on remove

Red-Black Trees

- Max height: $2 * \lg(n)$

- Constant number of rotations on insert (max 2), remove (max 3).

Summary of Balanced BST

Pros:

- Running Time:
 - Improvement Over:
- Great for specific applications:

Summary of Balanced BST

Cons:

- Running Time:
- In-memory Requirement:

Red-Black Trees in C++

C++ provides us a balanced BST as part of the standard library:

```
std::map<K, V> map;
```

```
V & std::map<K, V>::operator[] ( const K & )
```

```
std::map<K, V>::erase( const K & )
```

Red-Black Trees in C++

```
iterator std::map<K, V>::lower_bound( const K & );  
iterator std::map<K, V>::upper_bound( const K & );
```

Every Data Structure So Far

	Unsorted Array	Sorted Array	Unsorted Linked List	Sorted Linked List	Binary Tree	BST	AVL
Find							
Insert							
Remove							
Traverse							