Data Structures
Review

CS 225
Brad Solomon

December 8, 2025

Good luck on your exams!

UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Department of Computer Science

Announcements

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Material covered here is not only material in class!

Represents only an attempt to provide some helpful resources.

Brad's suggested review strategies:

1) Go through lecture content (focus on review slides)

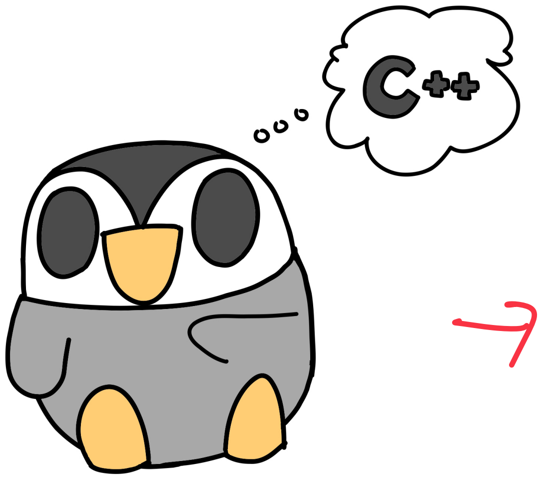
If there's material you don't remember fully, rewatch lecture

2) Go through practice exams once

If there's material you are struggling with, focus efforts here

3) Review course assignments

Don't look at your solution until after attempting it from scratch

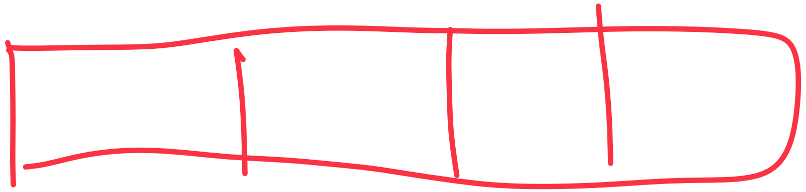


Exam 0 content

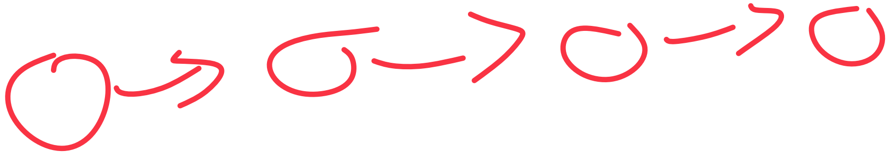
→ Look back at most assignments

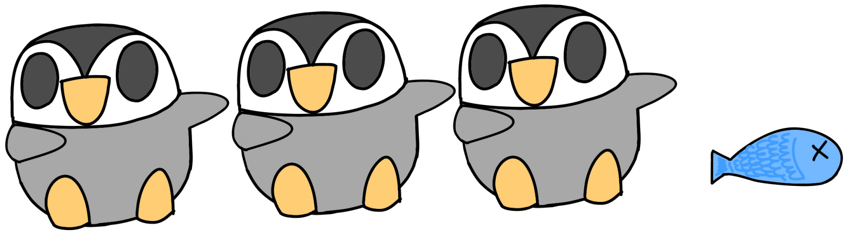
Lists

Array



Linked list





List Implementation

September 10 (Quack Lecture)

	Singly Linked List	Array
Look up **arbitrary** location ↳ random access	O(n)	O(1)
Insert after **given** element	O(1)	O(n)
Remove after **given** element	O(1)	O(n)
Insert at **arbitrary** location	Find is O(n) Mod is O(1) O(n)	Find is O(1) Mod is O(n) O(n)
Remove at **arbitrary** location	O(n)	O(n)
Search for an input **value**	O(n)	O(n)

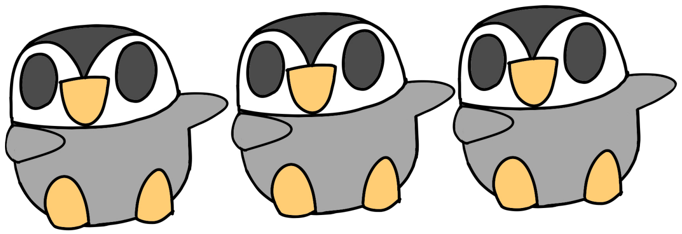
Special Cases:

insert Front (head)
remove

insert Back when not full
remove

Lists

November 7 (Review Lecture)



The not-so-secret underlying implementation for many things

O(1)* amortized

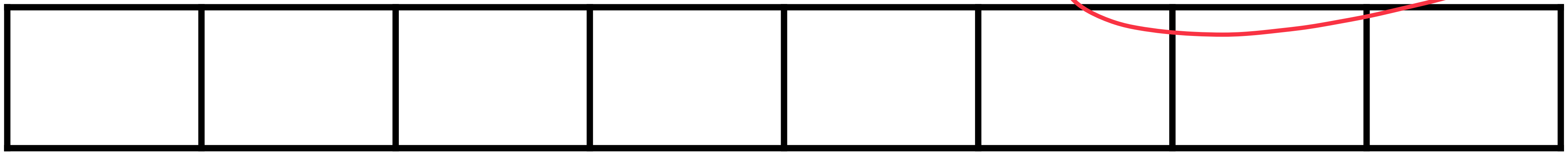
	Singly Linked List	Array
Look up **arbitrary** location	O(n)	O(1)
Insert after **given** element	O(1)	O(n)
Remove after **given** element	O(1)	O(n)
Insert at **arbitrary** location	O(n)	O(n)
Remove at **arbitrary** location	O(n)	O(n)
Search for an input **value**	O(n)	O(n)
Special Cases:	insertFront	insertBack (not full)

Stack and Queue

November 7 (Review Lecture)

Taking advantage of special cases in lists / arrays

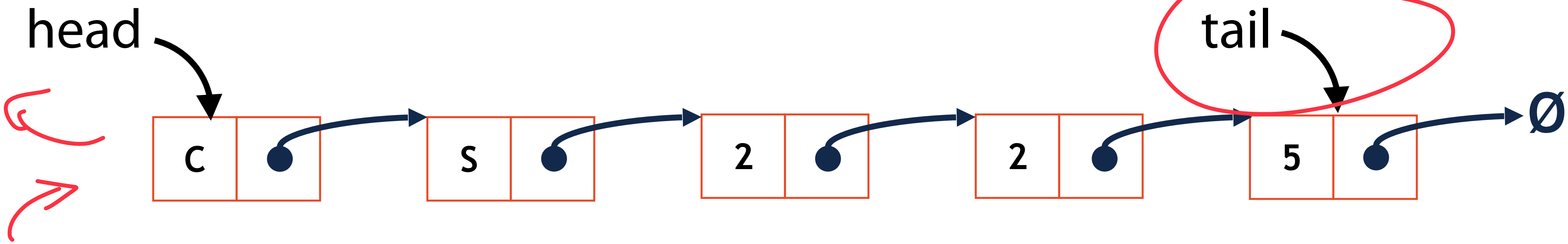
insertBack O(1)



queue

insertFront O(1)

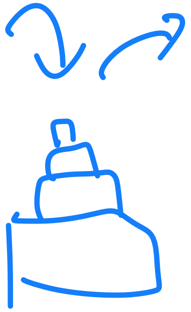
insertBack O(1)



Stack ADT

September 10 (Quacks Lecture)

- [Order]: Last in first out (LIFO)



- [Implementation]: Trivially as vector or LL

student Q

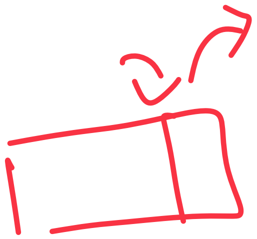
Front is top

- [Runtime]: $\boxed{O(1)^*}$

* if array is not full
if array is full, amortized still says O(1)

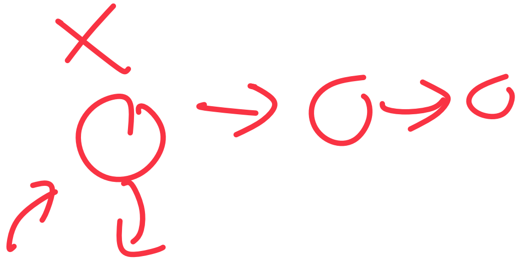
Stack ADT

- [Order]: LIFO (Last in first out)



- [Implementation]: Array (such as std::vector)

Linked List also works using insert / remove Front



- [Runtime]: O(1) Push and Pop

If using array, O(1)* if we need to resize.

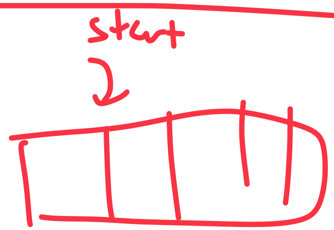
Queue ADT

- [Order]: First in first out
- [Implementation]: Trivially as LL
  w/ circular queue as array
- [Runtime]: $O(1)$ *
  when array amortized $O(1)$

September 12 (Queue-Iterator Lecture)

Queue ADT

- [Order]: FIFO (First In First Out)



- [Implementation]: Circular Queue as Array

Linked List also works using removeFront / insertBack or ~~vice versa~~

- [Runtime]: O(1)

Iterators

September 12 (Queue-Iterator Lecture)

The actual iterator is defined as a class **inside** the outer class:

1. It must be of base class **std::iterator**

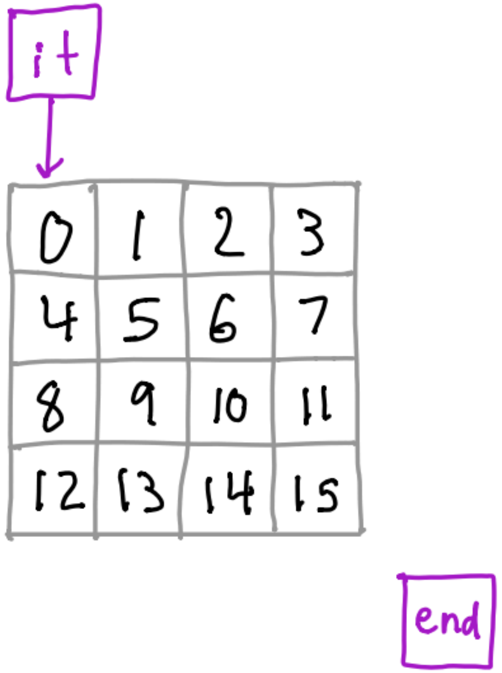
2. It must implement at least the following operations:

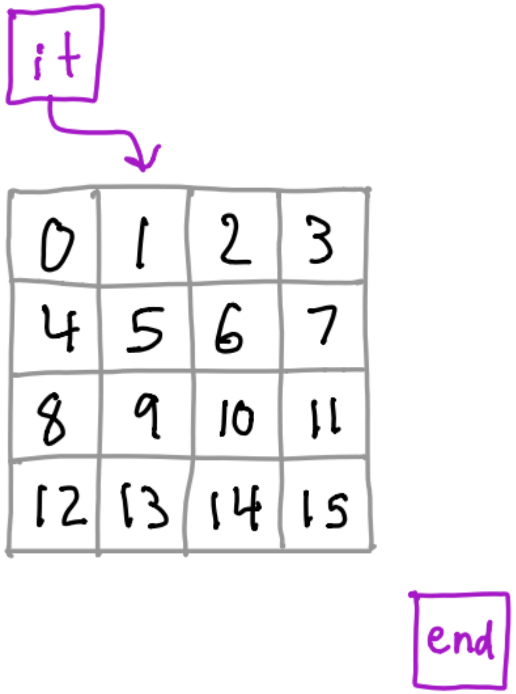
```
Iterator& operator ++()
```
get next

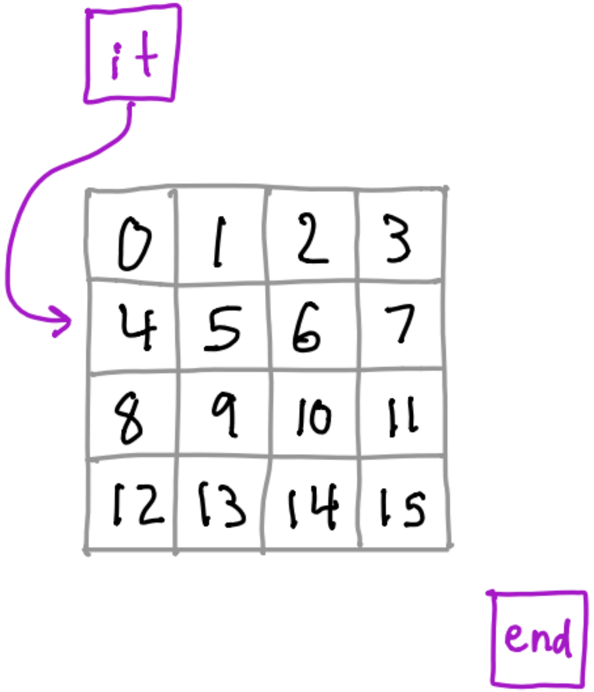
```
const T & operator *()
```
dereference

```
bool operator !=(const Iterator &)
```
compare iterator objects

Iterators (225 Webpage Resources)



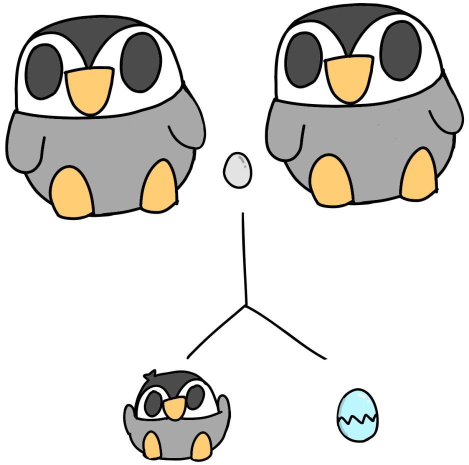




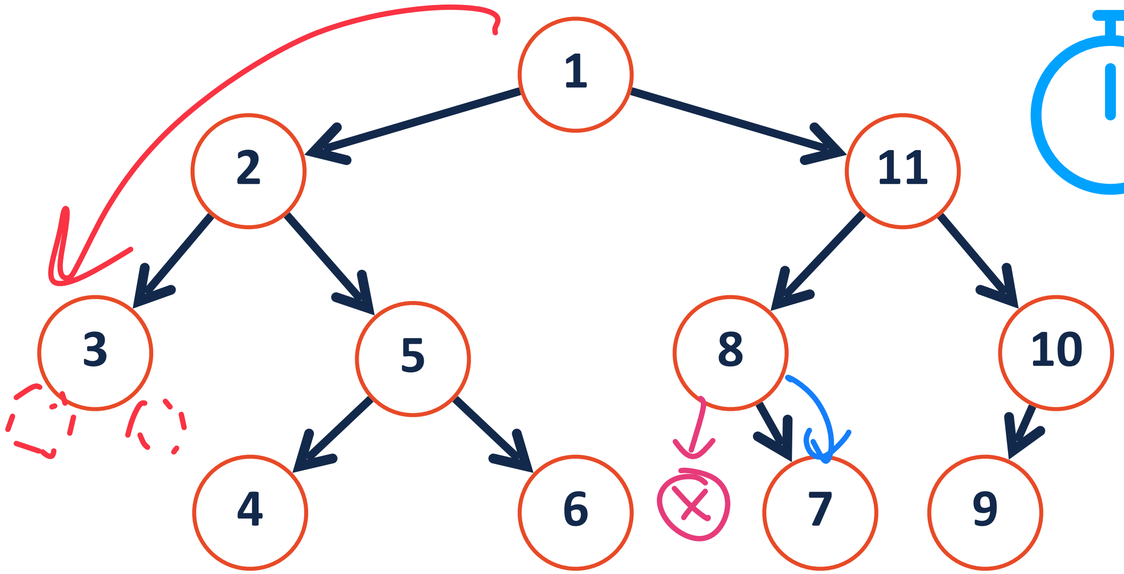
https://courses.grainger.illinois.edu/cs225/fa202~~4~~5/resources/iterators/

Trees

Labs in this section great for reviewing fundamentals



Tree Traversals



root

Pre-order: 1 2 3 5 4 6 11 8 7 10 9

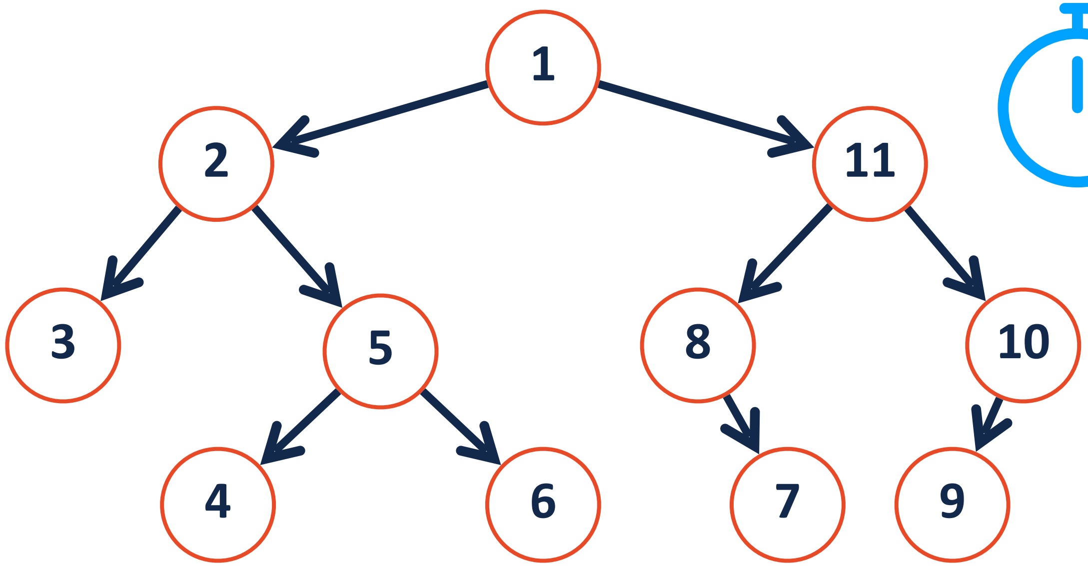
left most child

root

In-order: 3 2 4 5 6 1 8 7 11 9 10

Post-order: 3 4 6 5 2 7 8 9 10 11 1

Tree Traversals



Pre-order: 1, 2, 3, 5, 4, 6, 11, 8, 7, 10, 9

In-order: 3, 2, 4, 5, 6, 1, 8, 7, 11, 9, 10

Post-order: 3, 4, 6, 5, 2, 7, 8, 9, 10, 11, 1

Depth First Search

Max size of stack ≈ Height of Tree

Explore as far along one path as possible before backtracking

Make a stack initialized with root

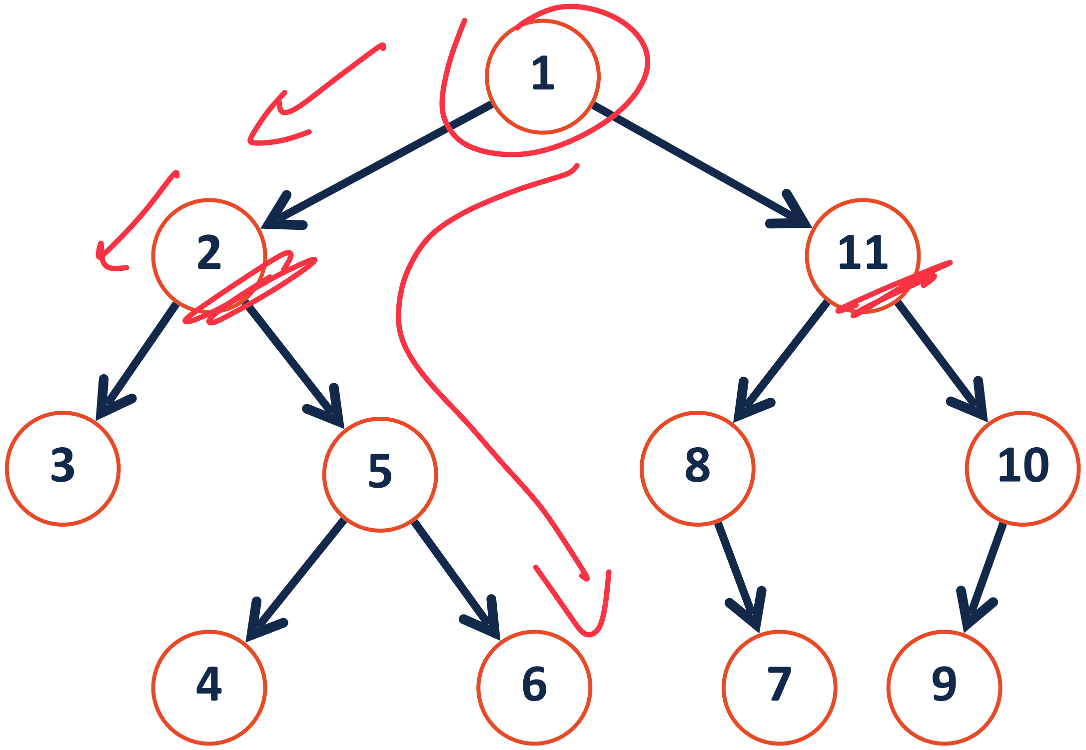
While stack isn't empty:

  Pop top element (as `tmp`)

  Print `tmp`

  Push `tmp->right` to stack

  Push `tmp->left` to stack



Stack: 1, 11, 2, 5, 3, 6, 4, 10, 8, 7, 9

Print: 1, 2, 3, 5, 4, 6, 11, 8, 7, 10, 9

Breadth First Search

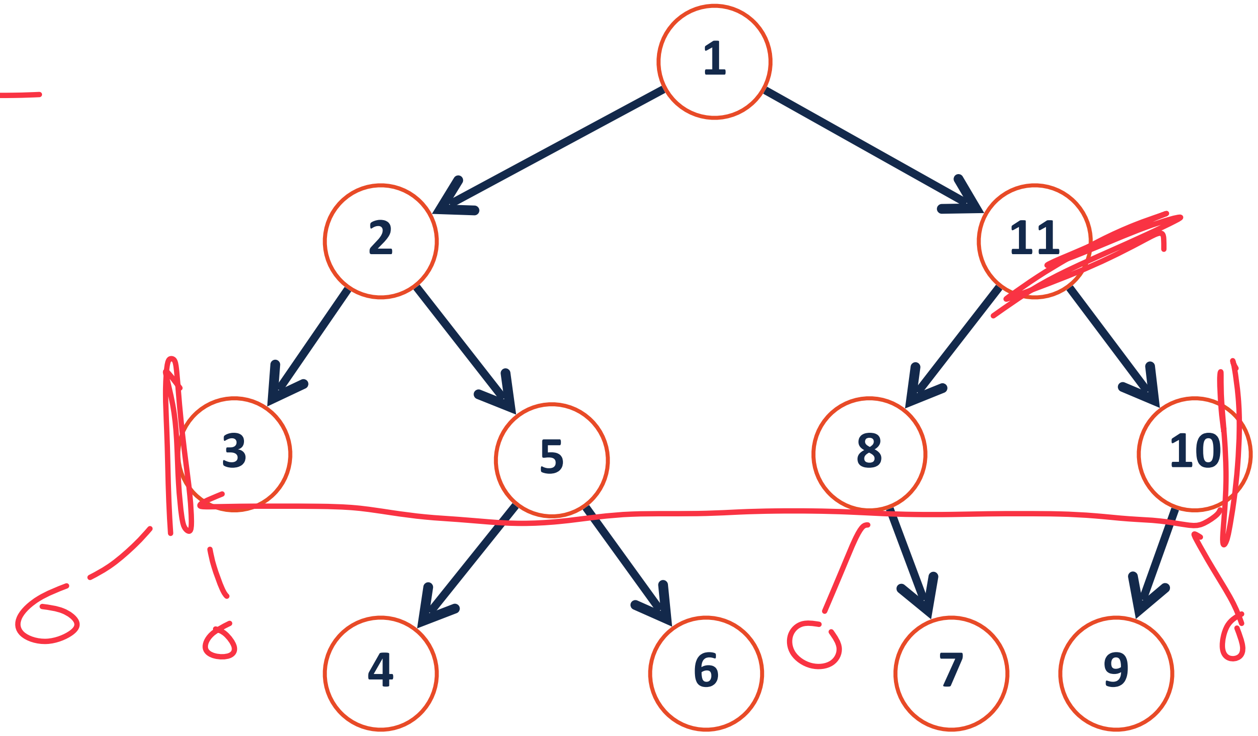
Max size of queue ≈ Width of Tree

Fully explore depth i before exploring depth i+1

Make a queue initialized with root

While queue isn't empty:

- Dequeue front element (as `tmp`)
- Print `tmp`
- Enqueue `tmp->left`
- Enqueue `tmp->right`



1 2 11

Queue: 1, 2, 11, 3, 5, 8, 10, 4, 6, 7, 9

Print: 1, 2, 3, 5, 4, 6, 11, 8, 7, 10, 9

BST Find

`find(66)`

A recursive function based around value of root:

Base Case: If root is `null`, return root

Let `tmp = root->key()`

`tmp == query`, return root

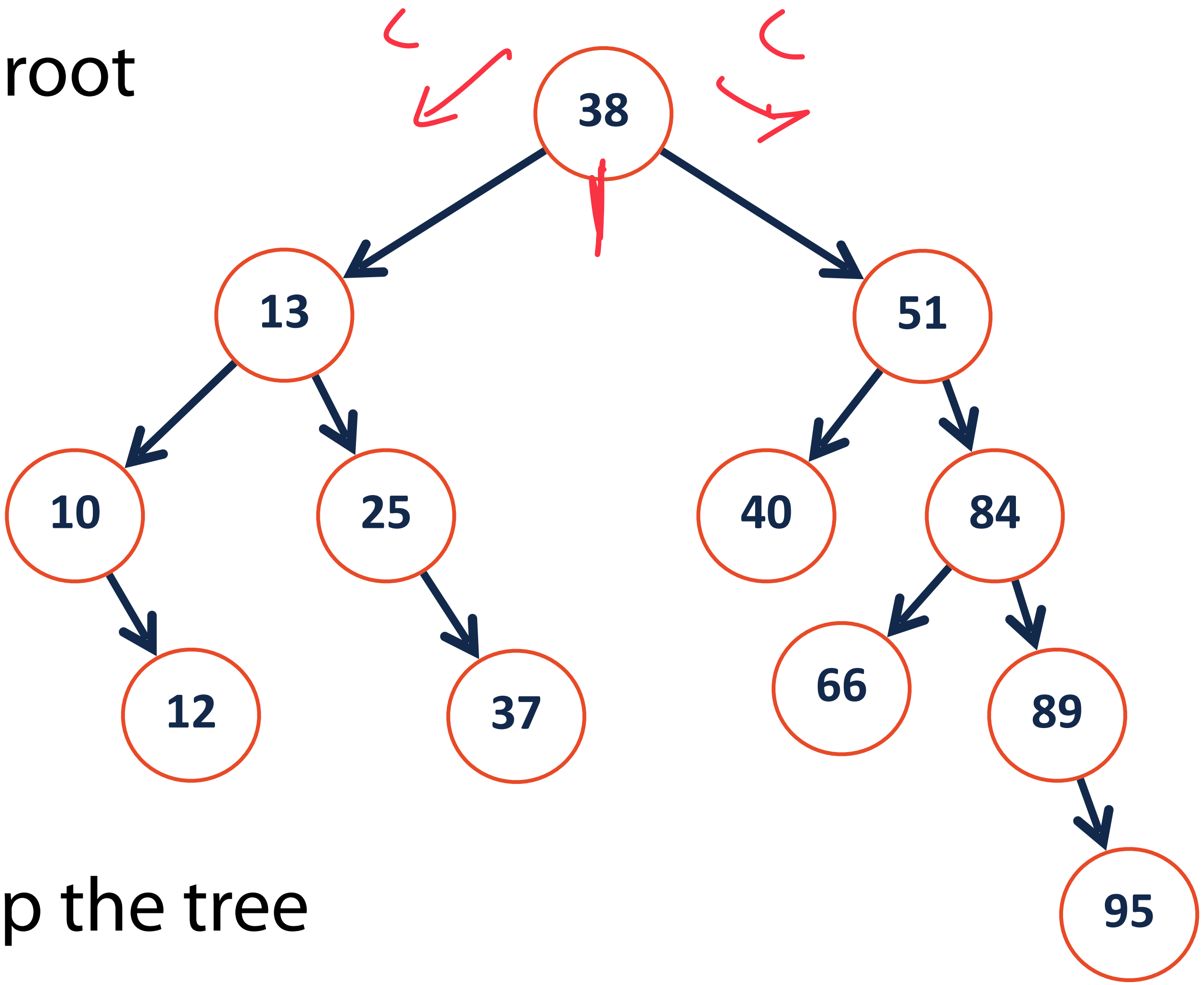
Recursion:

`tmp < query`, recurse right

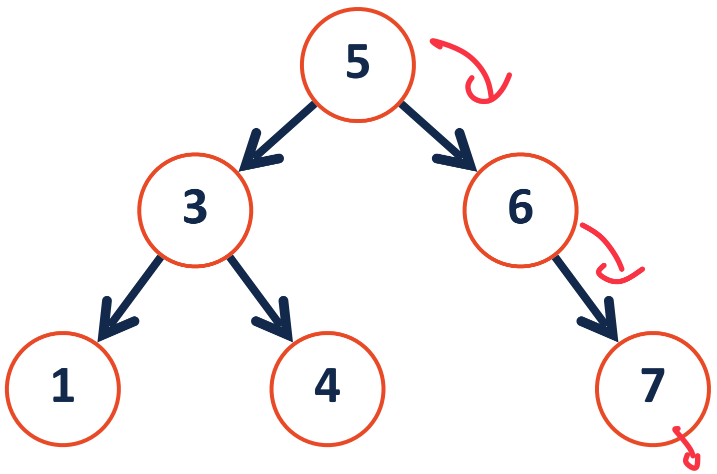
`tmp > query`, recurse left

Combining:

Return the recursive value back up the tree



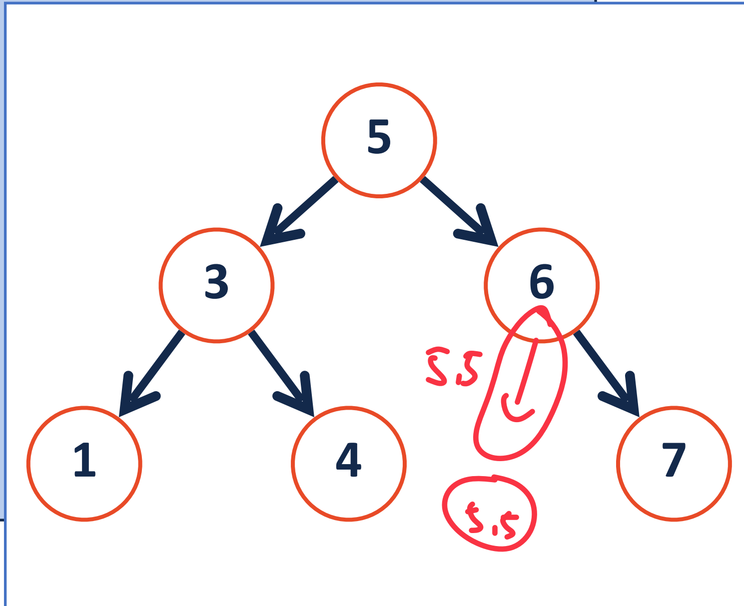
```
template<typename K, typename V>

______TreeNode *&________ _find(TreeNode *& root, const K & key) {


// Base Case
if(root == nullptr || root->key == key){
    return root;
}

// Recursive Step ("Combining step" is 'return')
if (root->key > key){
    return _find(root->left, key);
}

return _find(root->right, key);


}
```



```
template<typename K, typename V>

void _insert(const K & key, const V & val) {

    return _insert(root, key, val);
}
```

```
template<typename K, typename V>

void _insert(TreeNode *& root, const K & key, const V & val) {

TreeNode *& tmp = _find(root, key);


tmp = new treeNode(key, val);




}
```

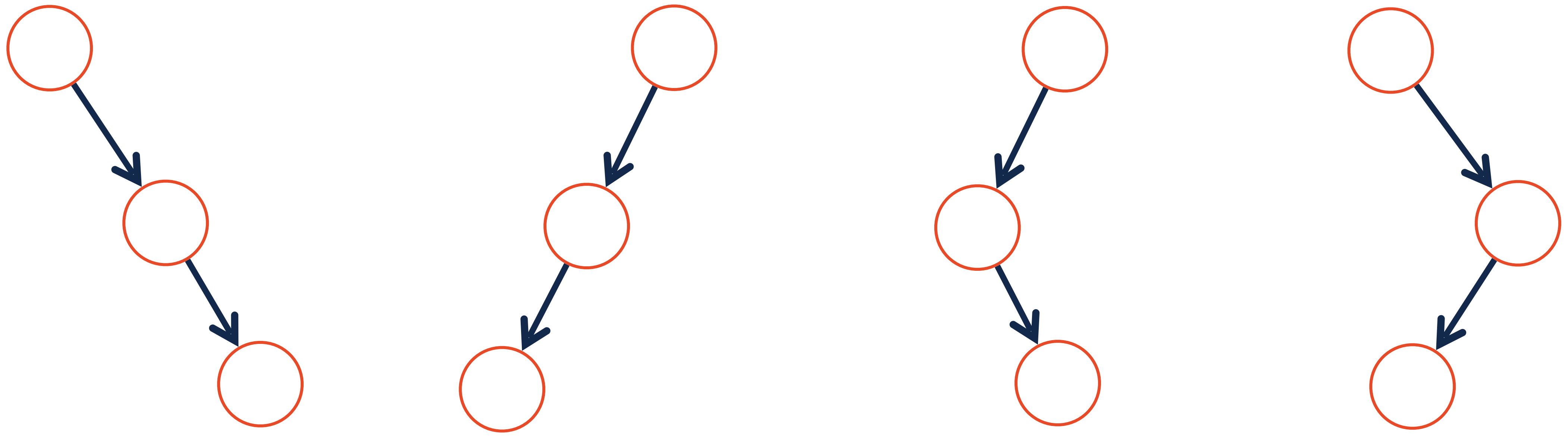
Reference to pointer!



BST Analysis – Running Time

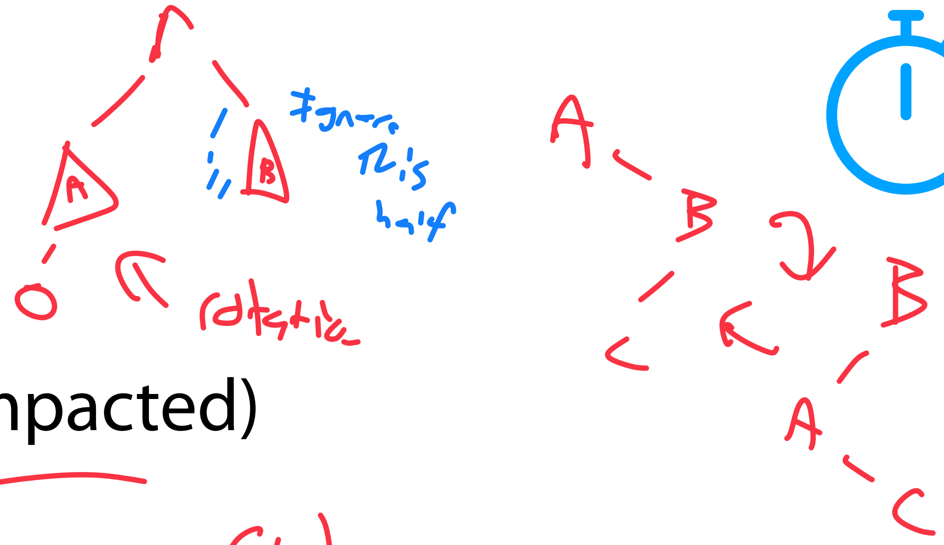
Operation	BST Worst Case
find	O(h) = O(n)
insert	O(h) = O(n)
remove	O(h) = O(n)
traverse	O(n)

AVL Rotations



	Left	Right	LeftRight	RightLeft
Root Balance:	2	-2	-2	2
Child Balance:	1	-1	1	-1

AVL Rotations



Four kinds of rotations: (L, R, LR, RL)

other

1. All rotations are local (subtrees are not impacted)

2. The running time of rotations are constant $O(1)$

3. The rotations maintain BST property

Goal: AVL tree will be balanced

↳ this will make height bounded by $\log(n)$

AVL Tree Analysis

Guarantee:

1) Tree is balanced

For an AVL tree of height h:

Find runs in: $O(h)$.

Insert runs in: $O(h)$.

Remove runs in: $O(h)$.

Claim: The height of the AVL tree with n nodes is: $O(\log n)$.

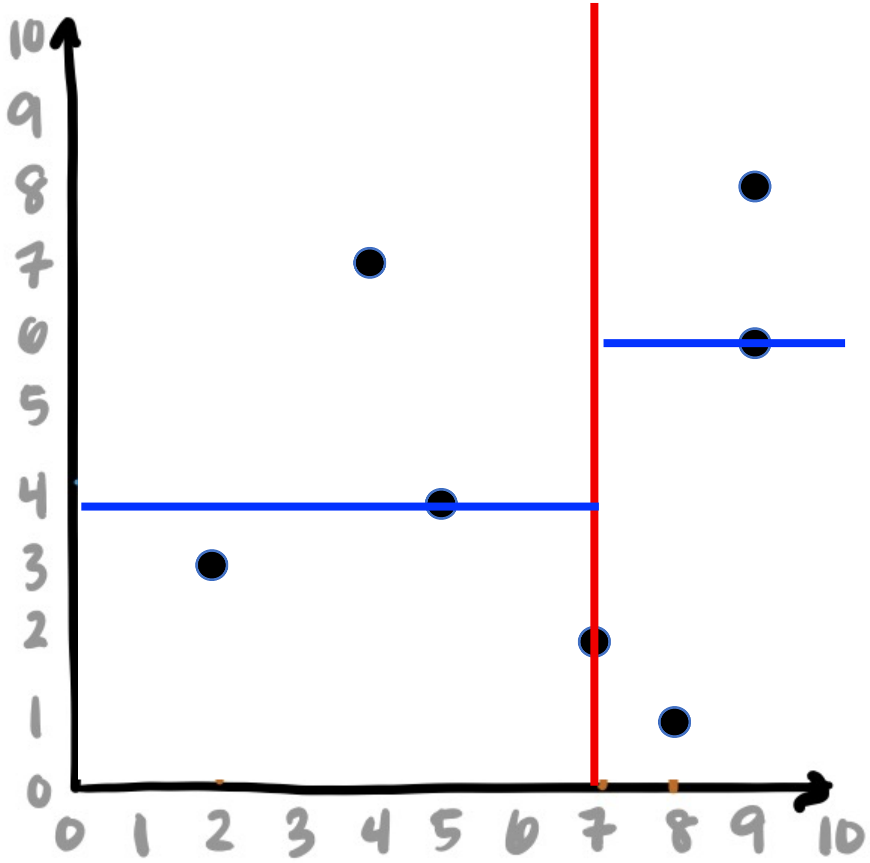
Nearest Neighbor: k-d tree

A **k-d tree** is similar but splits on points:

(7,2), (5,4), (9,6), (4,7), (2,3), (8,1), (9,8)

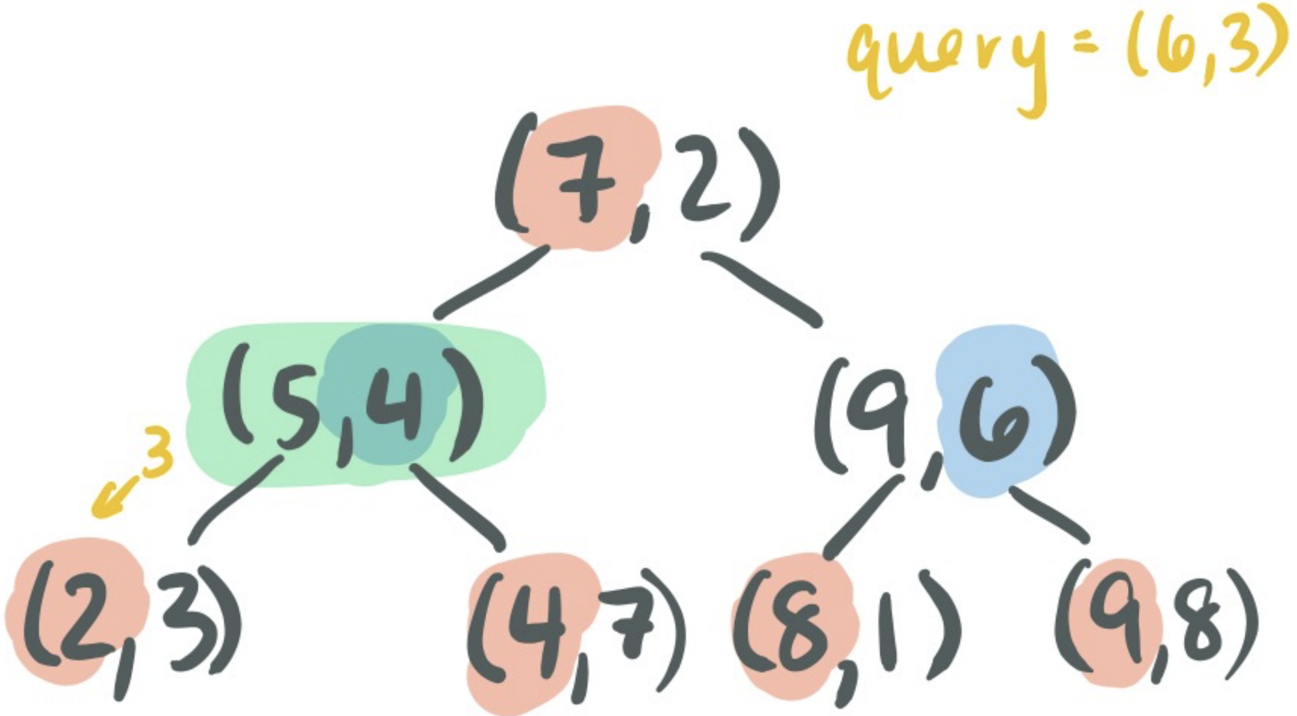
x-split (7,2)

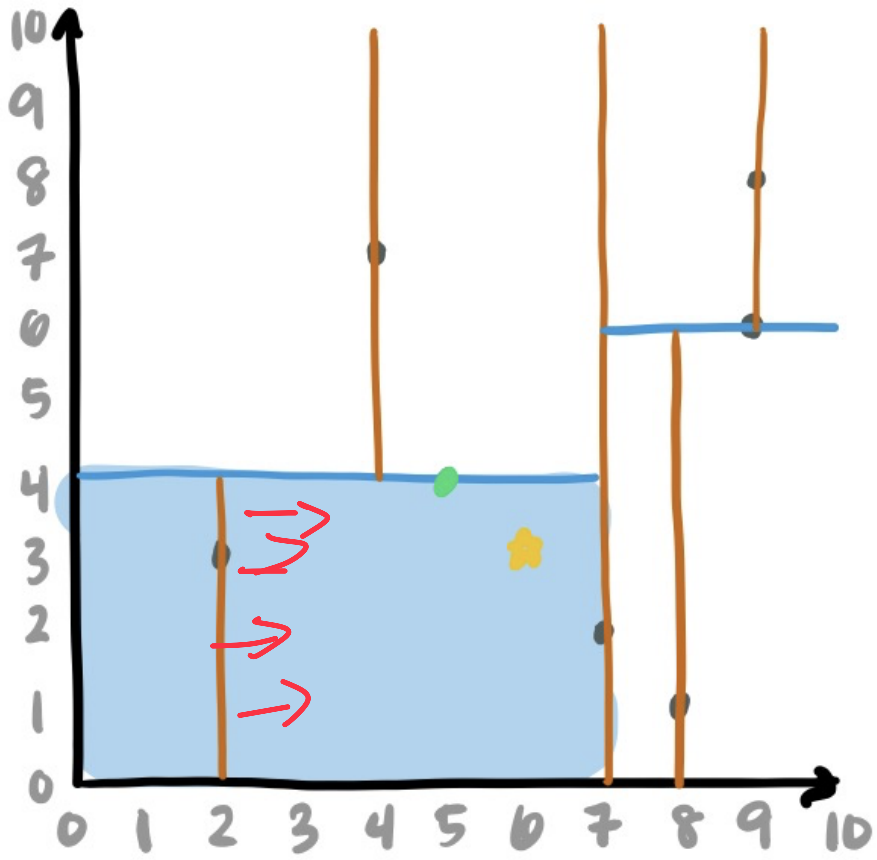
y-split (5,4) (9,6)



Nearest Neighbor: k-d tree

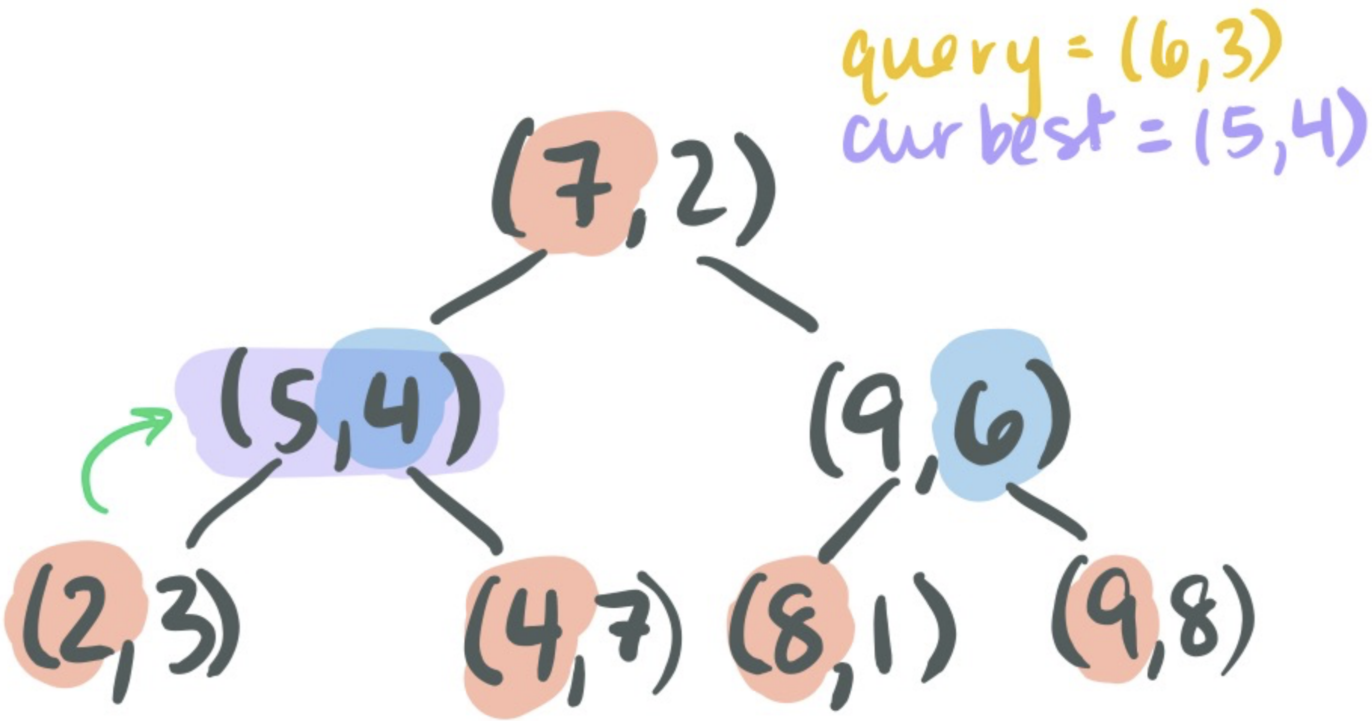
Search by comparing query and node in single **alternating** dimension

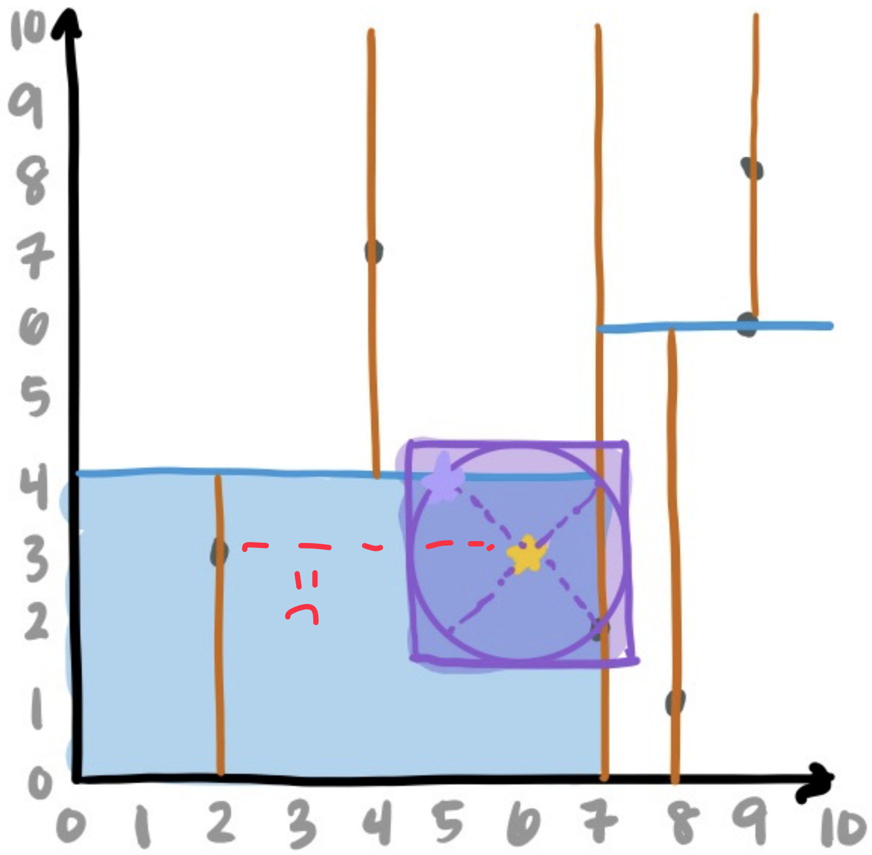




Nearest Neighbor: k-d tree

Backtracking: start recursing backwards -- store “best” possibility as you trace back

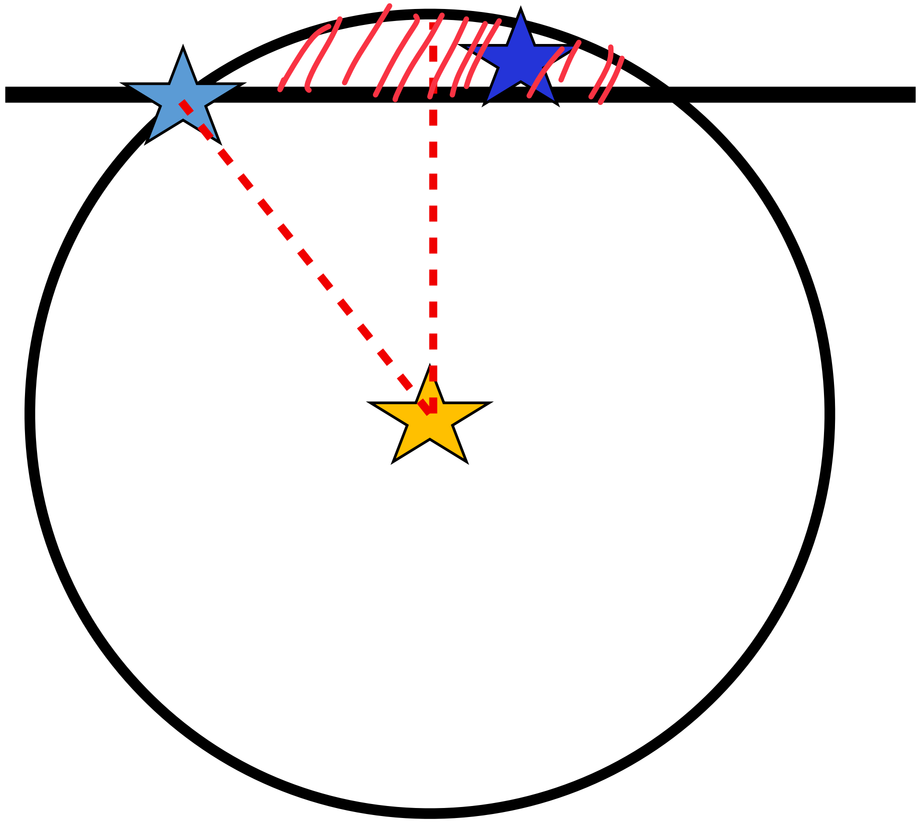


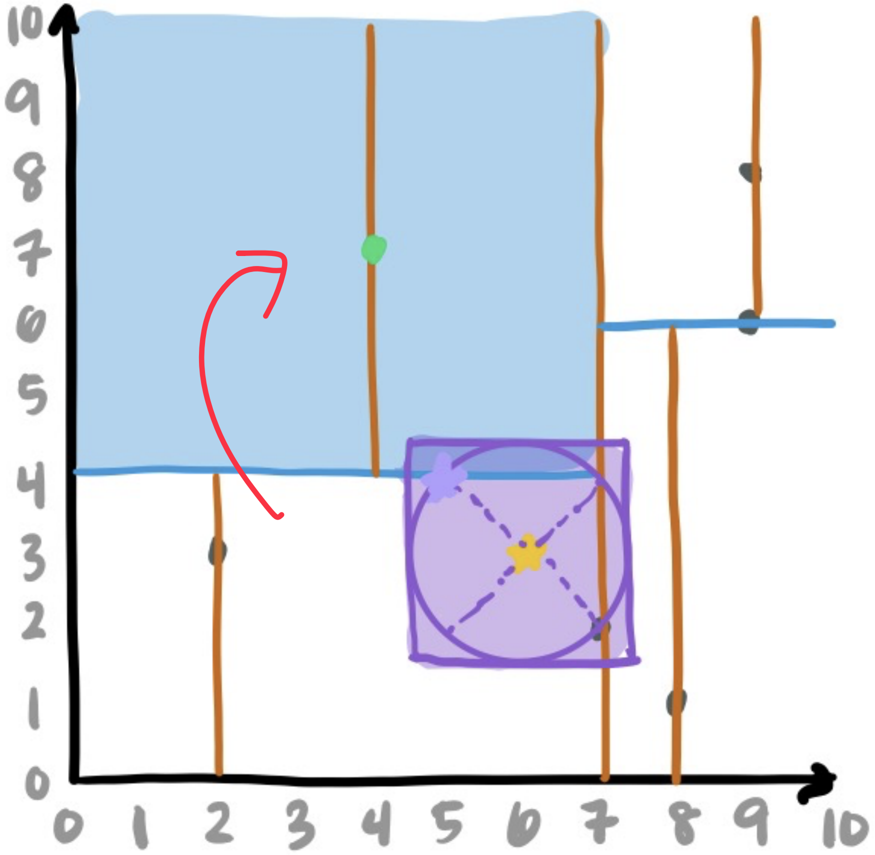


Nearest Neighbor: k-d tree

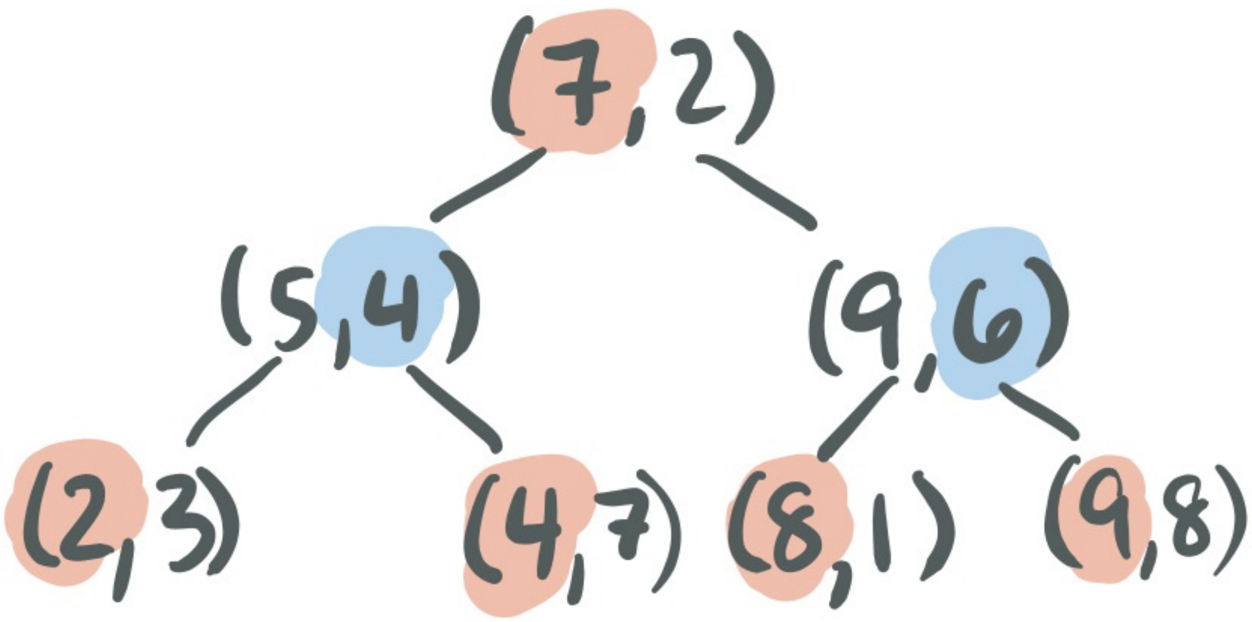
May have to recursively check other branches of tree — **why?**

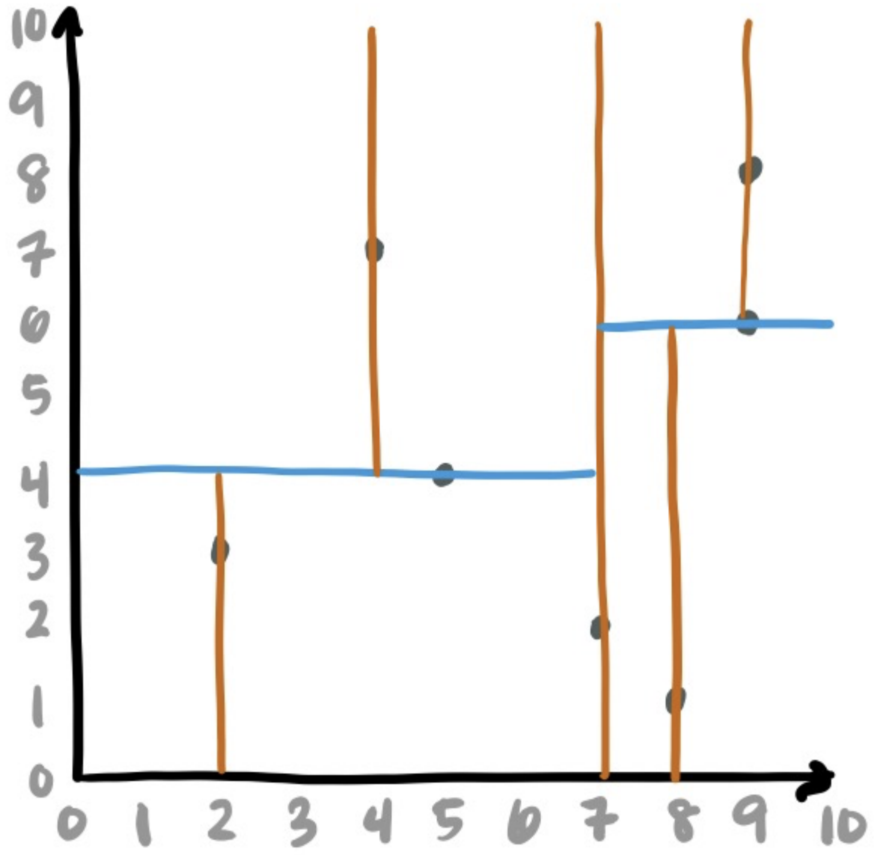
Potentially better point in this small area





Nearest Neighbor: k-d tree





BTree Properties

A **BTrees** of order **m** is an m-ary tree and by definition:

- All keys within a node are ordered
- All nodes contain no more than **m-1** keys.
- All internal nodes have exactly **one more child than keys**

→ 0 children

Root nodes can be a leaf or have ___ $[2, m]$ ___ children.

If int $(\frac{m}{2})$ is keys
+1 is children

All non-root, internal nodes have ___ $[\lceil \frac{m}{2} \rceil, m]$ ___ children.

All leaves in the tree are at the same level.

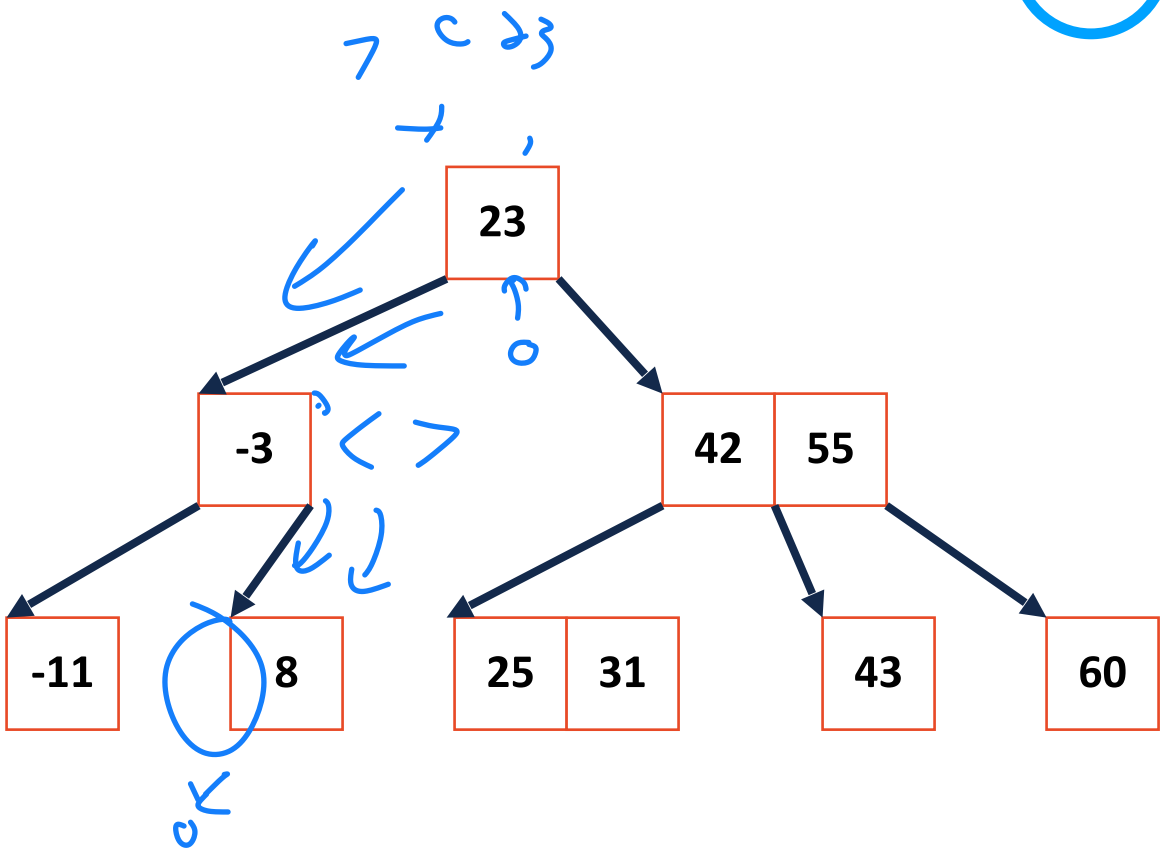
BTree Find

Find(7)

Base Case:

If root is empty, return

If leaf, do array find() and return



Recursive Step:

Array find() for match or first greater value

Recurse on appropriate child

Tip: Index of first greater value is index of child we want to visit!

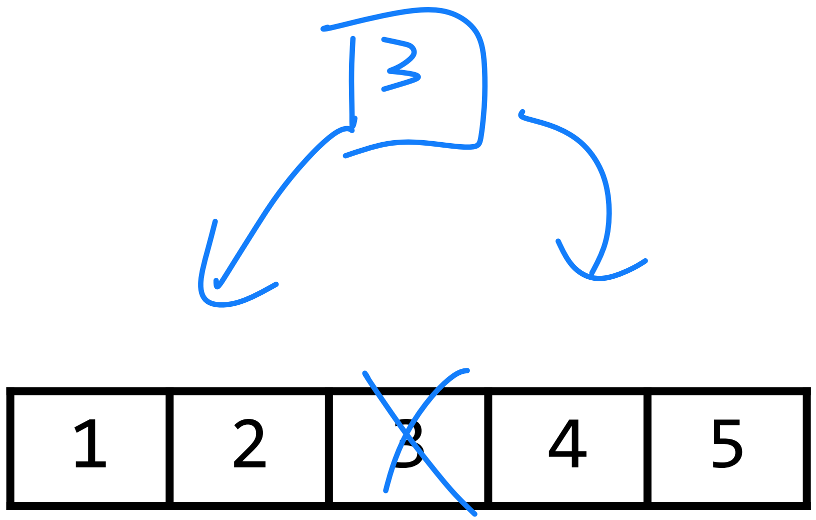
BTree Insertion

M = 5

When we hit **M** items, split into three nodes!

1) Create new parent node w/ median value

2) Split existing array into ~M/2 partitions

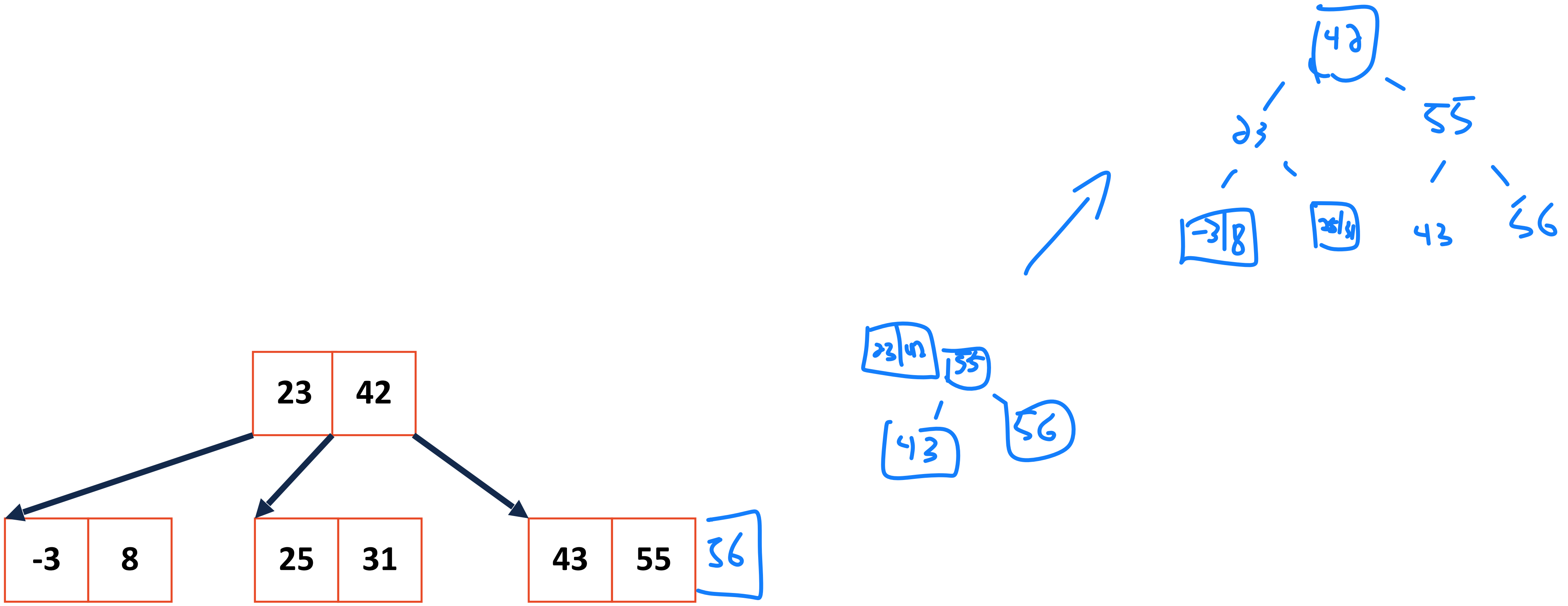


`Insert(1)`
`Insert(2)`
`Insert(3)`
`Insert(4)`
`Insert(5)`
`Insert(6)`
`Insert(7)`
`Insert(8)`

BTree Recursive Insert

Insert(56), M = 3

Insert always starts at a leaf but can propagate up repeatedly.



Final thoughts on Trees

Trees have a large space of **possible coding questions**

We hit **tree iterators** multiple times…

You saw **tree constructors of unusual shapes**…

You've seen trees on previous exams…

Heap

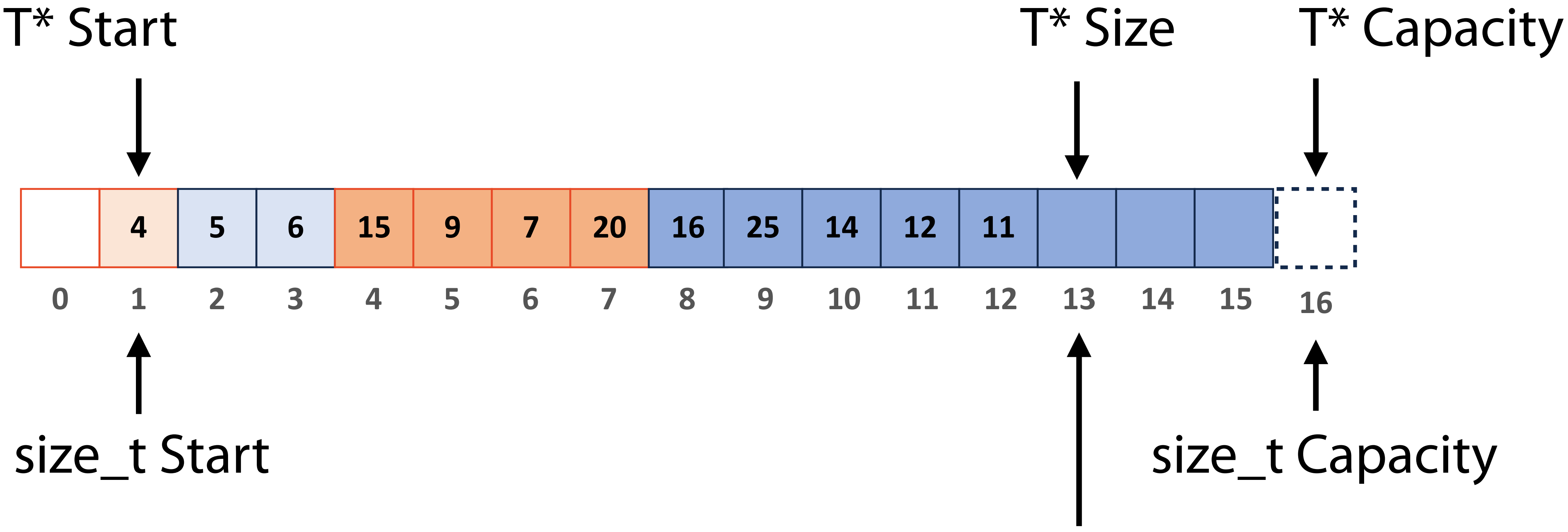
Taking advantage of special cases in lists / arrays

O(1) lookup

O(1) swap

O(1)* insertBack

Array List (Pointer implementation)



Array List (Index implementation)

(min)Heap

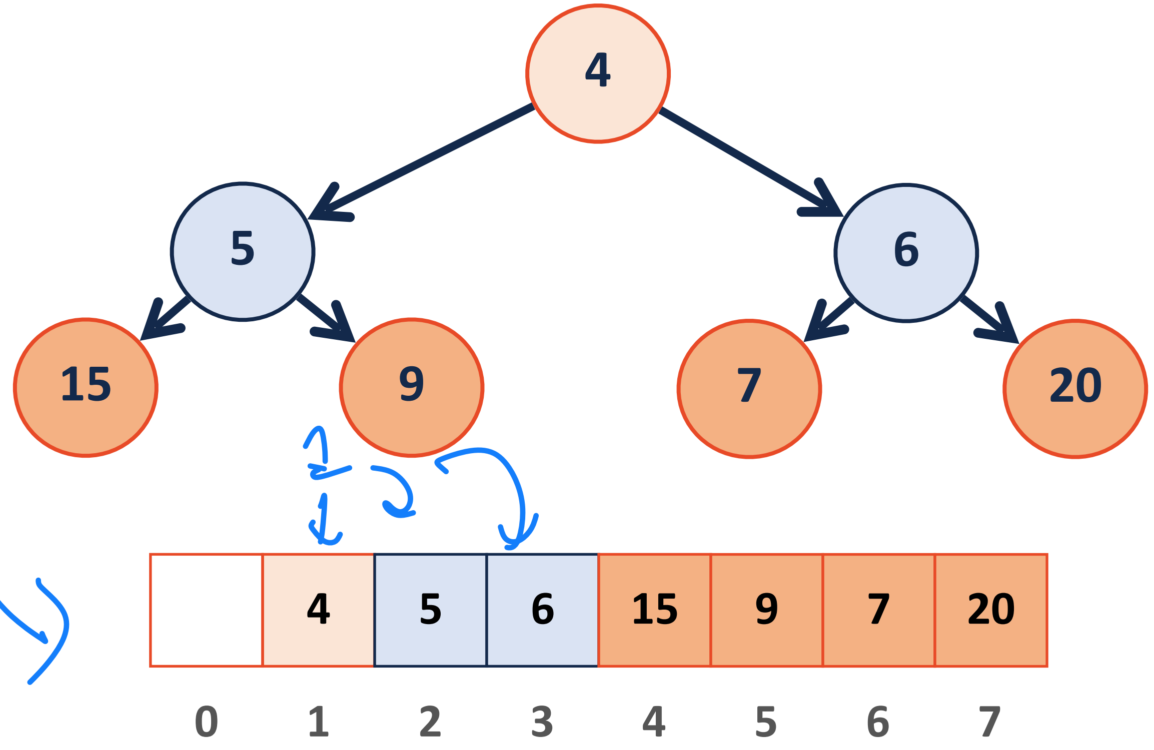
By storing as a complete tree, can avoid using pointers at all!

If index starts at 1:

`leftChild(i): 2i`

`rightChild(i): 2i+1`

`parent(i): floor(i/2)`



insert

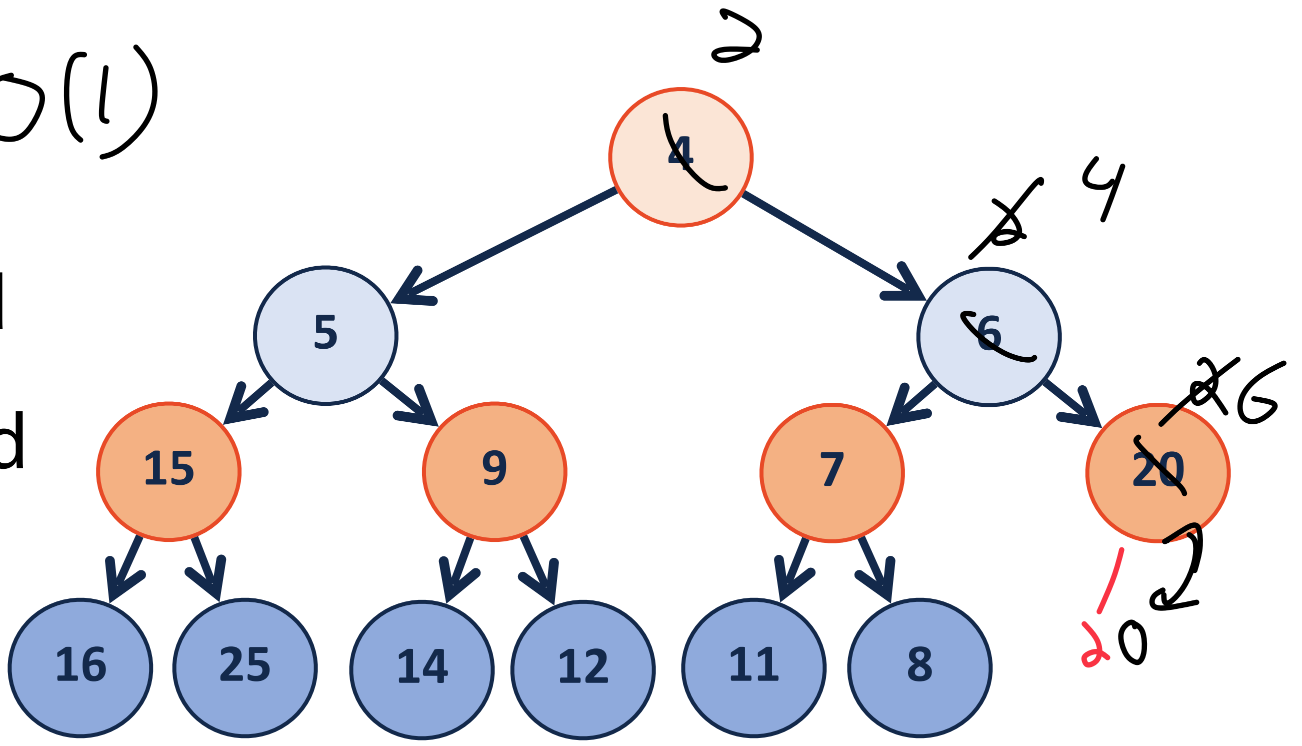
1) Insert at end of array

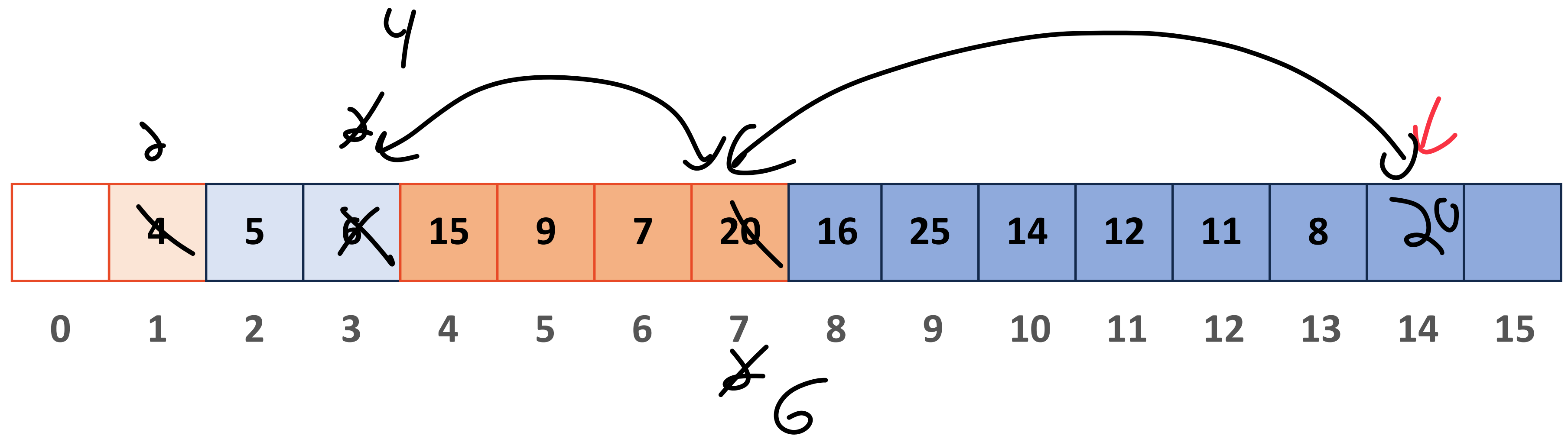
O(1)

2) Check if minHeap still valid

3) Swap with parent if needed

Steps 2 and 3 are recursive!





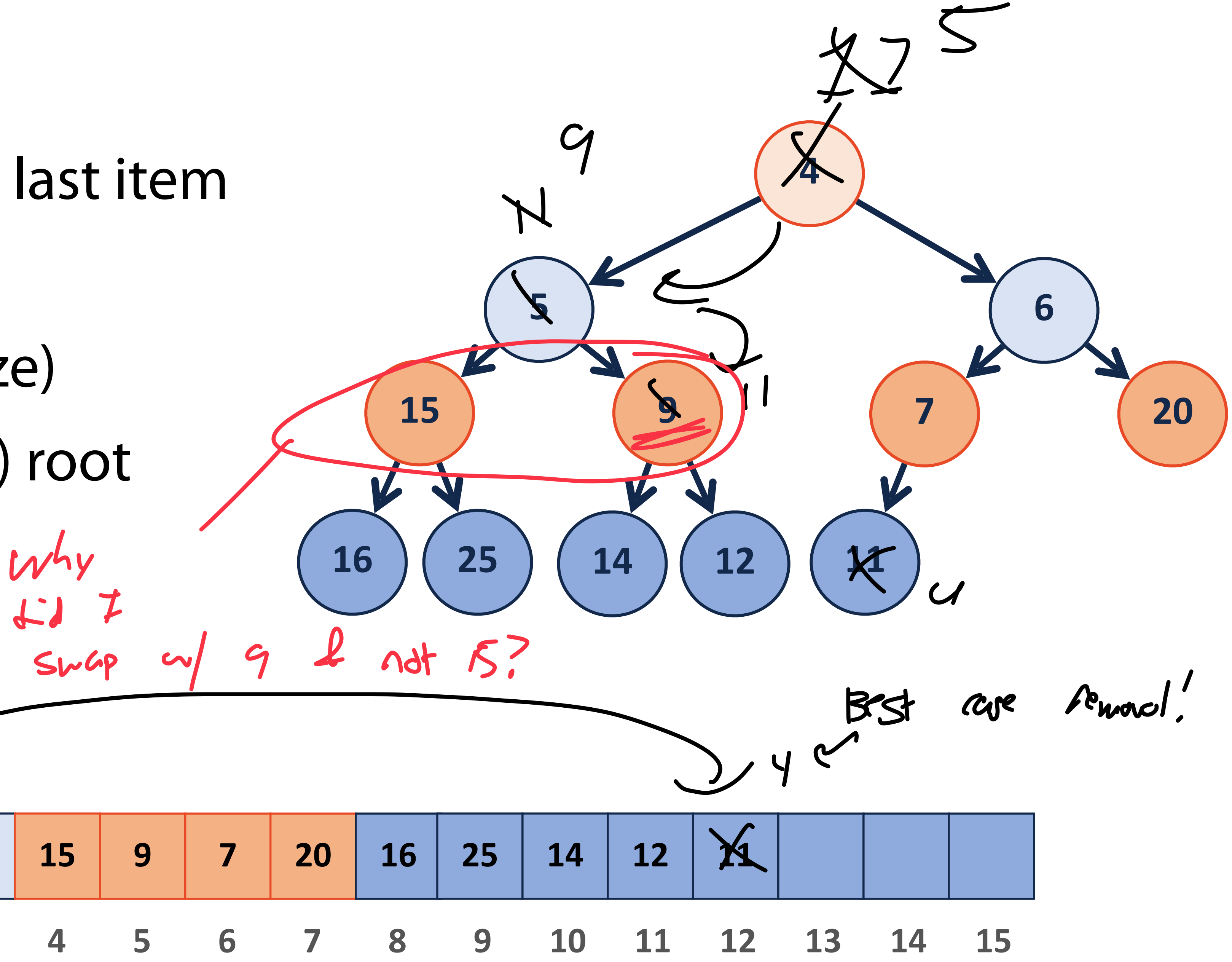
removeMin

1) Swap root with last item

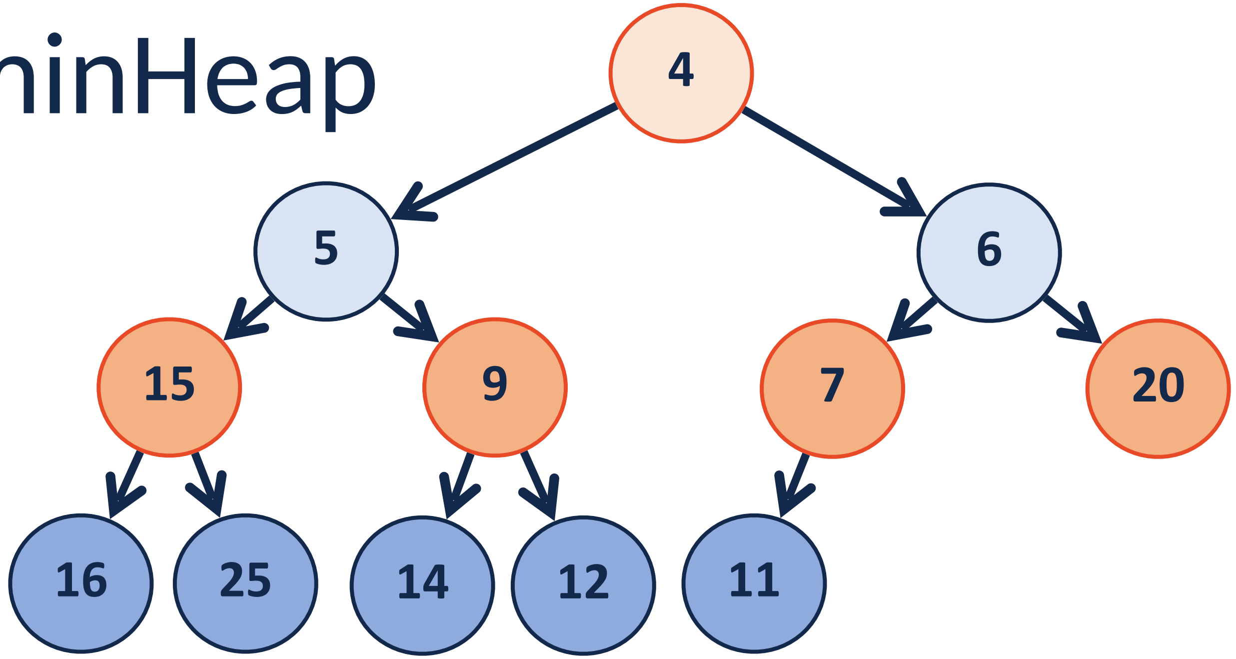
(and remove)

(and modify size)

2) HeapifyDown( ) root



minHeap



1. **Construction**

   O(n) :)

2. **Insert**

   O(log n)

3. **RemoveMin**

   O(log n)

	4	5	6	15	9	7	20	16	25	14	12	11				

minHeap is a good example of tradeoffs:

Array memory locality

Fast access of min item

Not intended for random access

Some wasted space

Final thoughts on Heaps

Building a heap on different datasets is a useful exercise

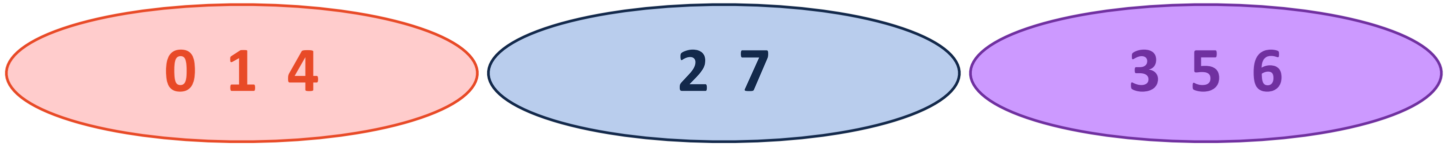
You haven’t been tested on heaps yet…

Disjoint Sets

Disjoint Set Implementation

Taking advantage of array lookup operations

Store an UpTree as an array, canonical items store **height** / **size**



0	1	2	3	4	5	6	7
-2	0	-2	-2	0	3	3	2
-3		-2	-3				

Find(k): Repeatedly look up values until **negative value**

Union(k_1, k_2): Update ***smaller*** canonical item to point to larger

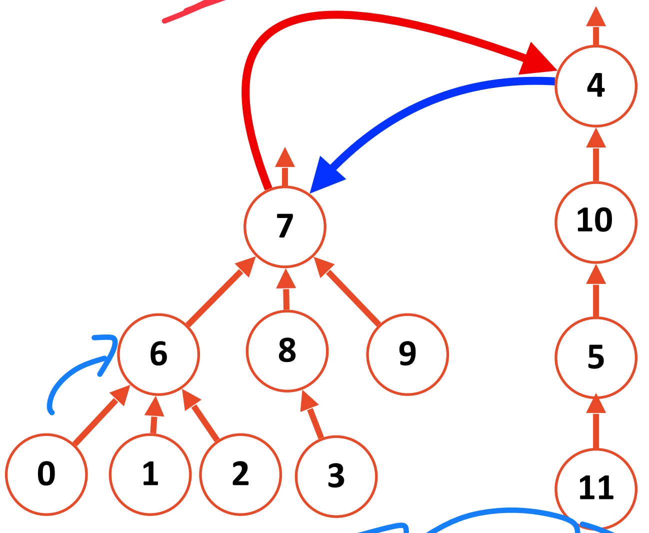
Update value of remaining canonical item

Disjoint Sets – Smart Union

Height 2 -> Height 3

Two O(1) methods of combining two sets

Claim: Both limit height to: O(log n).



Size 8 <- Size 4

Before Union / **After Union**

Union by height

4	…	7
-4		-3

4	…	7
-4		4

__Idea__: Keep the height of the tree as small as possible.

Union by size

4	…	7
-4		-8

4	…	7
7		-12

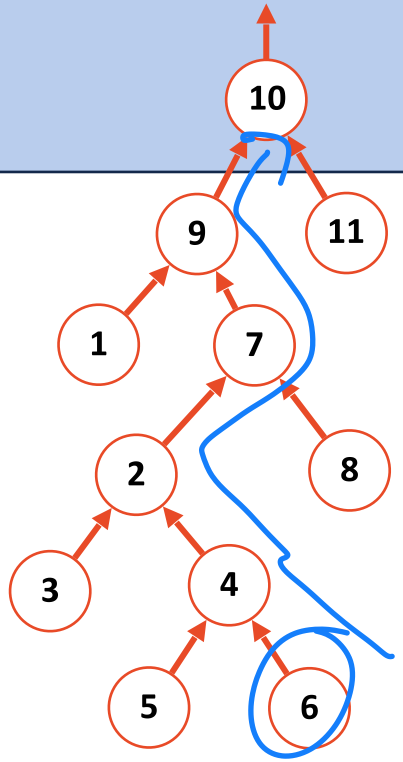
__Idea__: Minimize the number of nodes that increase in height

Disjoint Sets Path Compression

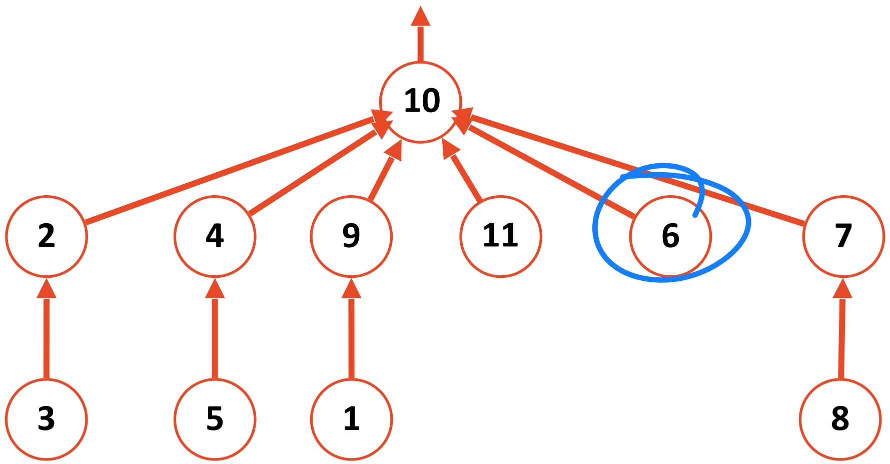
Find(6)

Minimizing number of O(1) operations

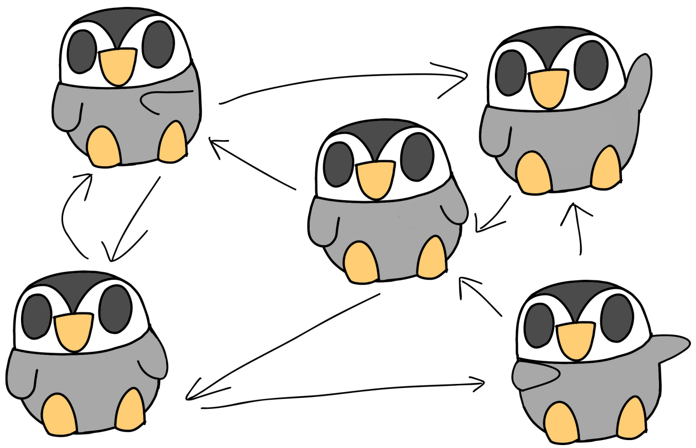
```
int DisjointSets::find(int i) {
  if ( s[i] < 0 ) { return i; }
  else {
    int root = find( s[i] );
    s[i] = root;
    return root;
  }
}
```



Yet another benefit to array usage!

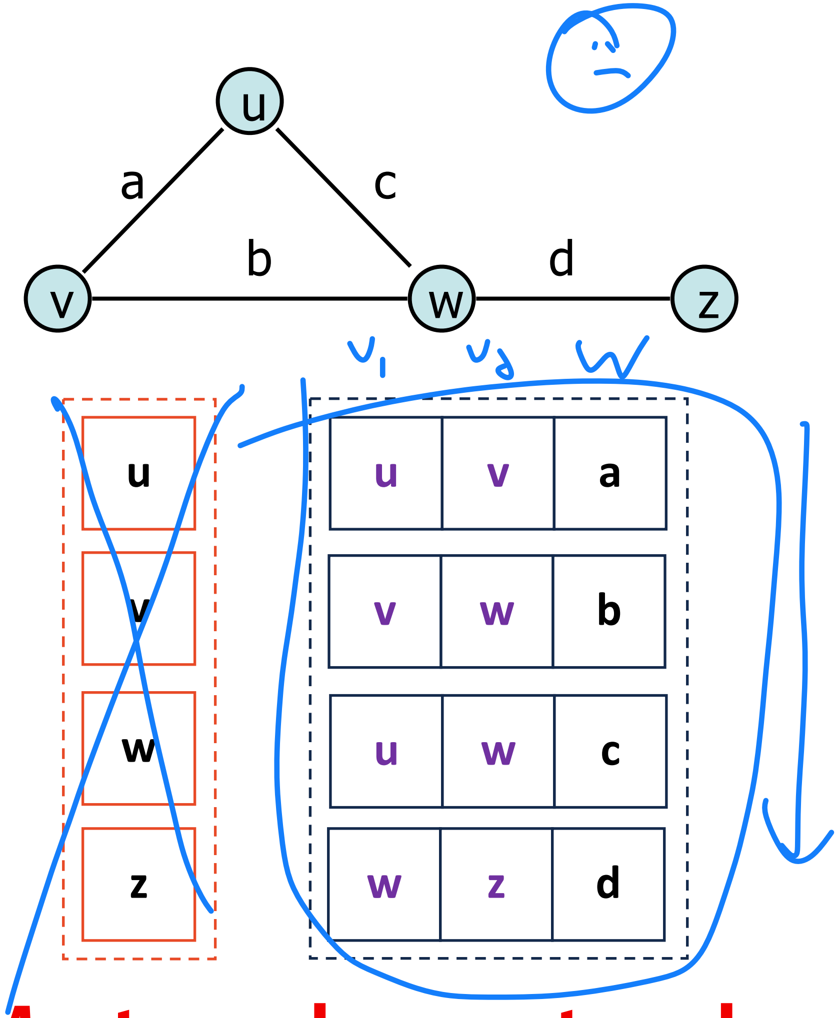


Graphs



Graph Implementation: Edge List $|V| = n, |E| = m$

The equivalent of an 'unordered' data structure



Vertex Storage:

An optional list of vertices

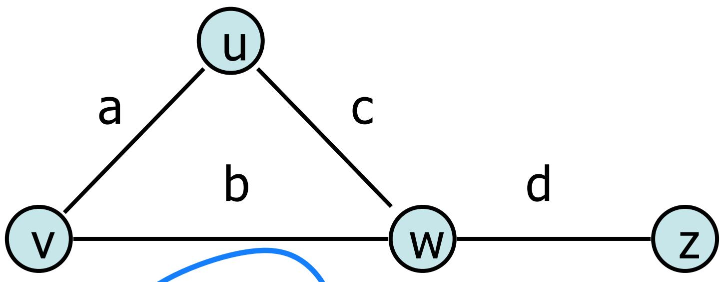
Edge Storage:

A list storing edges as (V1, V2, Weight)

Most graphs are stored as just an edge list!

Graph Implementation: Adjacency Matrix

$|V| = n, |E| = m$



u	0
v	1
w	2
z	3

	0	1	2	3
0	-	a	c	0
1		-	b	0
2			-	d
3				-

Vertex Storage:

A hash table of vertices

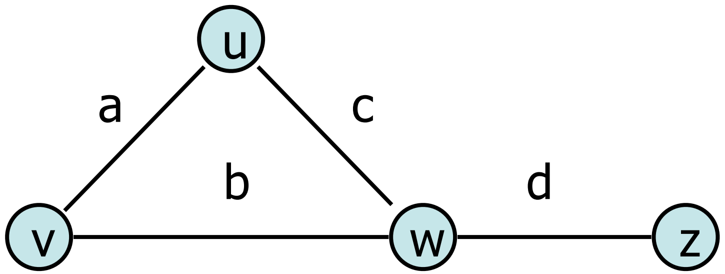
Implicitly or explicitly store index

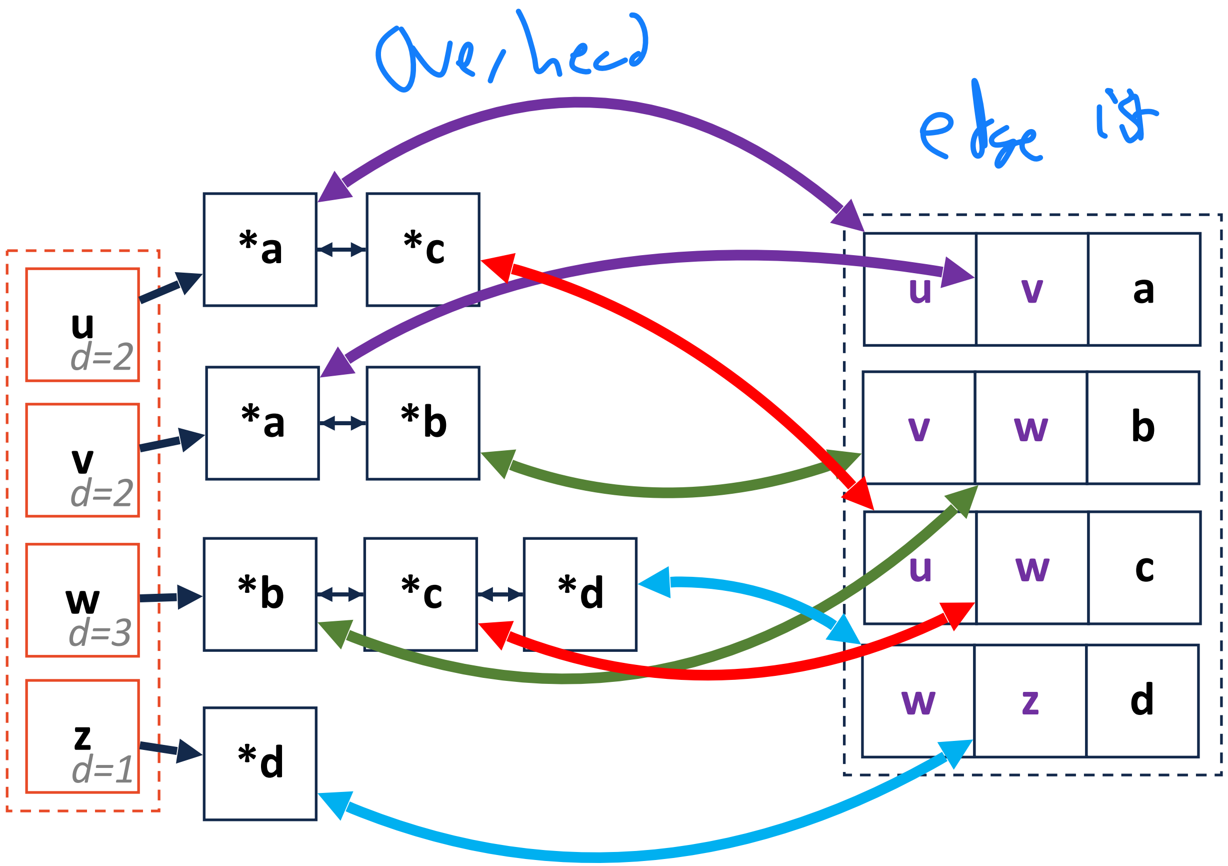
Edge Storage:

A |V| x |V| matrix of edges

Weight is stored at position (u, v)

Adjacency List





Vertex Storage:

A bidirectional linked list with size variable

Each node is a pointer to edge in edge list

Edge Storage:

A list of (v1, v2, weight) edges

Also store pointers back to nodes

Adjacency List

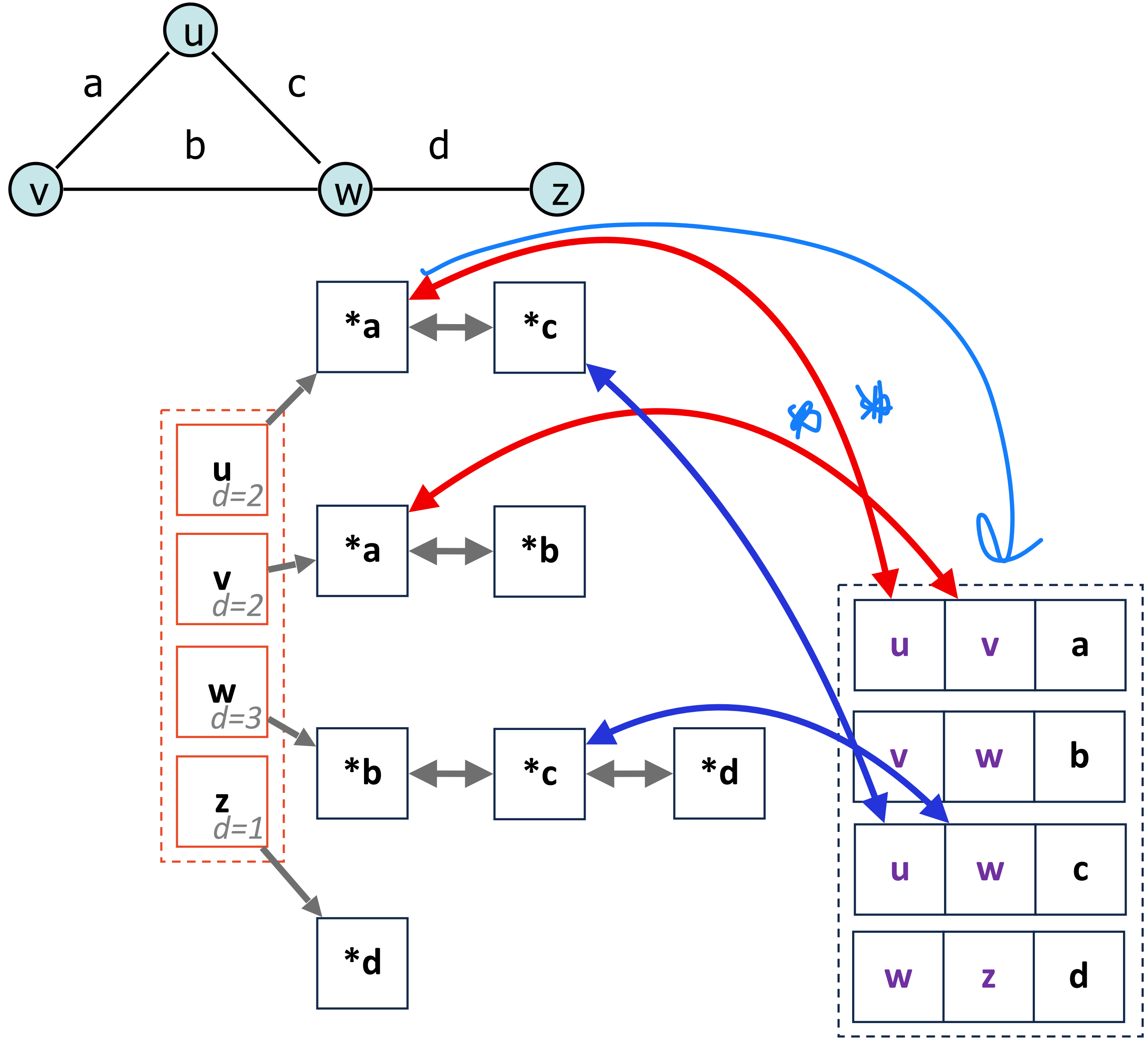
$|V| = n, |E| = m$

Adj List Node:

*Prev	*Edge	*Next

Edge List:

V1	V2	Weight
*V1	*V2	



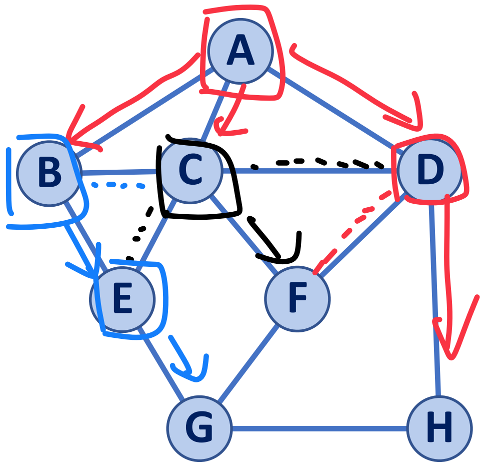
$|V| = n, |E| = m$

Expressed as O(f)	Edge List	Adjacency Matrix	Adjacency List
Space	n+m	n^2	n+m
insertVertex(v)	1*	n*	1*
removeVertex(v)	n+m	n	deg(v)
insertEdge(u, v)	1	1	1*
removeEdge(u, v)	m	1	min(deg(u), deg(v))
incidentEdges(v)	m	n	deg(v)
areAdjacent(u, v)	m	1	min(deg(u), deg(v))

$0 \le \deg(v) \le n-1$

Traversal: BFS

0) Initialize
↳ Queue (Put start in queue)
↳ Depth (start = 0)
↳ Predecessor (start = -1)



while queue not empty
↳ tmp = dequeue()
↳ Process all children
↳ Add to queue
↳ Set depth (tmp.depth +1)
↳ Set prev (to tmp)

→ Discovery (New vertex)
..... Cross (old vertex)

All discovery edges are in table
All edges not in table are cross

"visited" nodes have depth/pred

depth pred

v	d	P	Adjacent Edges
A	0	–	B C D
B	1	A	A C E
C	1	A	A B D E F
D	1	A	A C F H
E	2	B	B C G
F	2	C	C D G
G	3	E	E F H
H	2	D	D G

Check if vertex is "new"
Doesn't already have depth/pred

~~A~~ ~~B~~ ~~C~~ ~~D~~ ~~E~~ F H G

Front (queue) Back

October 29th Lecture

Traversal: BFS

$|V| = n, |E| = m$

Initialize queue / depth / predecessor

While queue not empty:

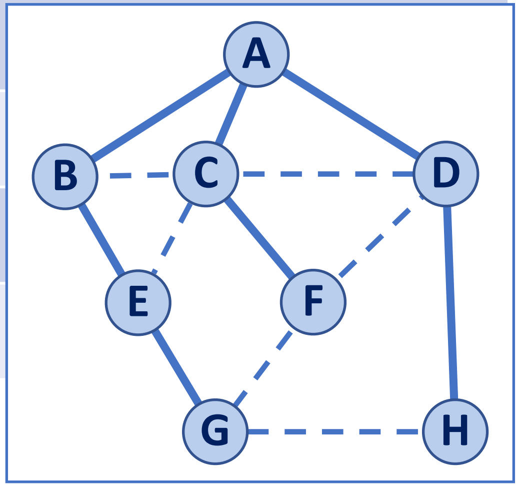
- Remove front vertex of queue
- Check if edge connects to new vertex
  - Set dist / pred if new vertex *
- Add unvisited edges to queue

like tree BFS but more on me

Every edge visited twice

Every vertex processed once

v	d	P	Adjacent Edges
A	0	-	B C D
B	1	A	A C E
C	1	A	A B D E F
D	1	A	A C F H
E	2	B	B C G
F	2	C	C D G
G	3	E	E F H
H	2	D	D G



~~A~~ ~~B~~ ~~C~~ ~~D~~ ~~E~~ ~~F~~ ~~H~~ ~~G~~

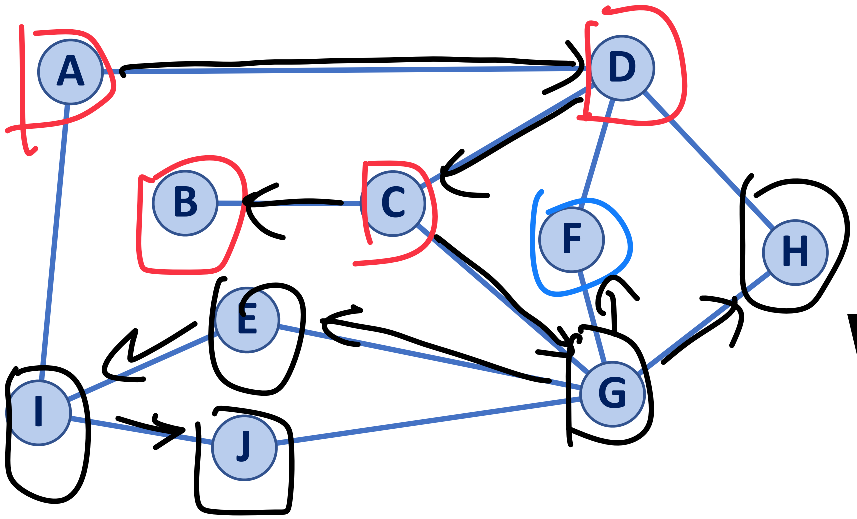
n^2

sparse $m \sim n$

dense $m \sim n^2$

Running time? O(n + m)

Traversal: DFS



Initialize dist / pred / stack

All dist null (start node dist 0)

All pred -1 (start node pred -1)

Stack loaded with start node

While stack not empty

tmp=stack.peek()

Process one child of tmp

Add to stack

dist = tmp.dist+1

pred = tmp

If no unvisited children

stack.pop()

~~H~~
~~F~~
~~J~~
~~I~~
~~E~~
G
~~B~~
C
D
A

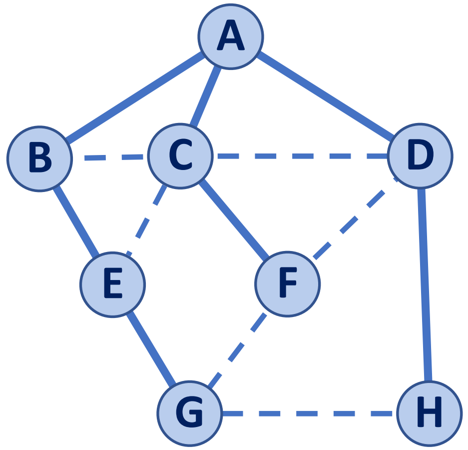
Stack

October 31st Lecture

Efficiency: DFS vs BFS

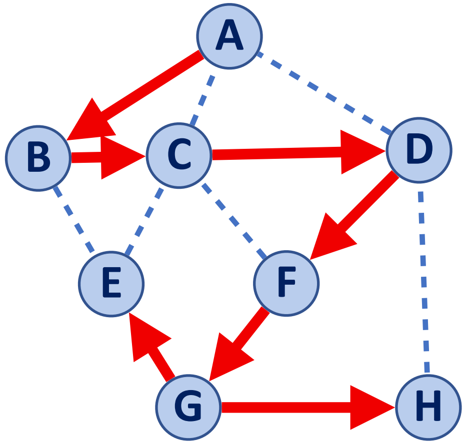
$|V| = n, |E| = m$

BFS: $O(n + m)$



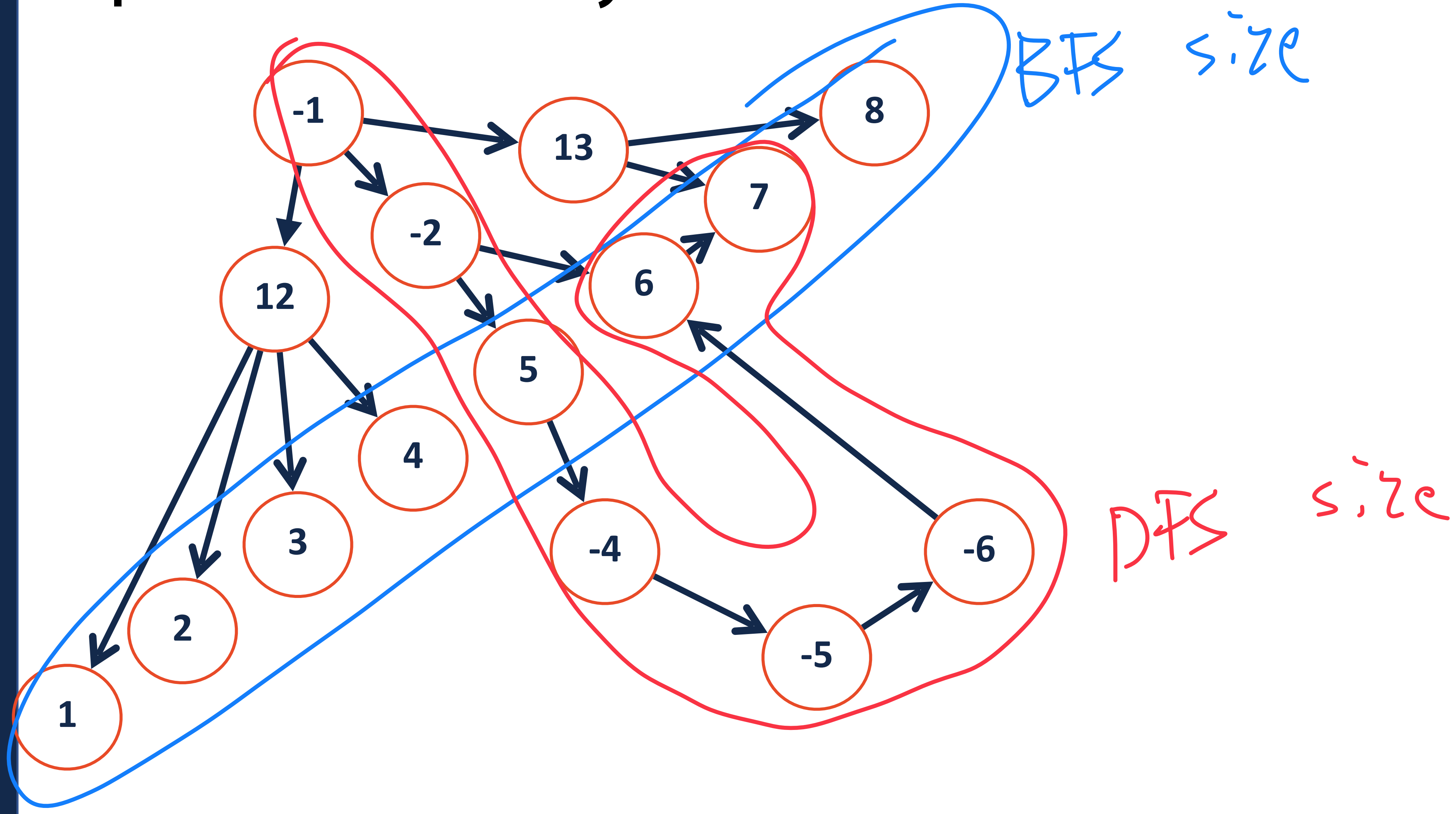
A B C D E F H G

DFS: $O(n + m)$



A B C D F G E H

Space Efficiency: DFS vs BFS



Summary: DFS and BFS

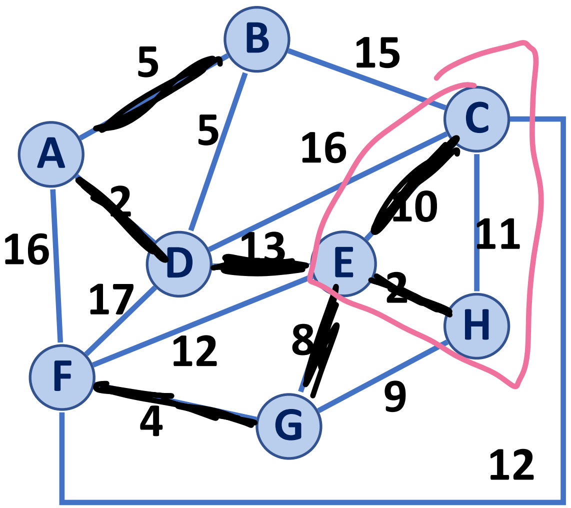
$|V| = n, |E| = m$

Both are **O(n+m)** traversals! They label every edge and every node

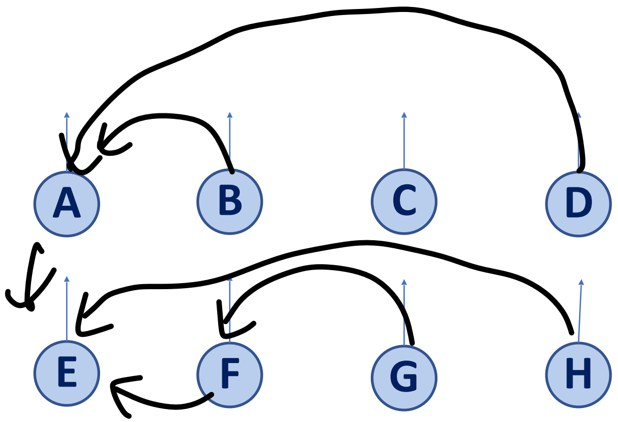
BFS	**DFS**
Solves unweighted MST	Solves unweighted MST
Solves shortest path	
Solves cycle detection	Solves cycle detection
Memory bounded by width	Memory bounded by longest path

Kruskal's Algorithm

(A, D) ✓
(E, H) ✓
(F, G) ✓
(A, B) ✓
(B, D) ✗
(G, E) ✓
(G, H) ✗
(E, C) ✓
(C, H) ✗
(E, F) ✗
(F, C) ✗
(D, E) ✓
(B, C)
(C, D)
(A, F)
(D, F)



At timestep here



November 3rd

1) Build a **priority queue** on edges

A minheap

or

A sorted array

2) Build a **disjoint set** on vertices

All vertices start as their own set

3) Loop through min edges

If edge connects two disjoint sets

update *Union sets and record edge in MST*

4) Stop when:

N-1 edges recorded we have MST

Only a single disjoint set remains

Kruskal's Algorithm

```
1  KruskalMST(G):
2    DisjointSets forest
3    foreach (Vertex v : G.vertices()):
4      forest.makeSet(v)
5
6    PriorityQueue Q    // min edge weight
7    Q.buildFromGraph(G.edges())
8
9    Graph T = (V, {})
10
11   while |T.edges()| < n-1:
12     Vertex (u, v) = Q.removeMin()
13     if forest.find(u) != forest.find(v):
14        T.addEdge(u, v)
15        forest.union( forest.find(u),
16                      forest.find(v) )
17
18   return T
19
```

$|V| = n, |E| = m$

What is the Big O?

2 — 4: O(n)

6 — 7: Heap: O(m)
Sorted List: O(m log m)

11: m x <12-17>

12—17: Heap: O(log m)
Sorted List: O(1)

Disjoint set we treat as O(1) b/c path compression w/ smart union

Kruskal's Algorithm

Priority Queue:	Heap	Sorted Array
Building :7	O(m)	O(m log m)
Each removeMin :12	O(m log m)	O(m)

Both result in m + m log m

Why is heap good?

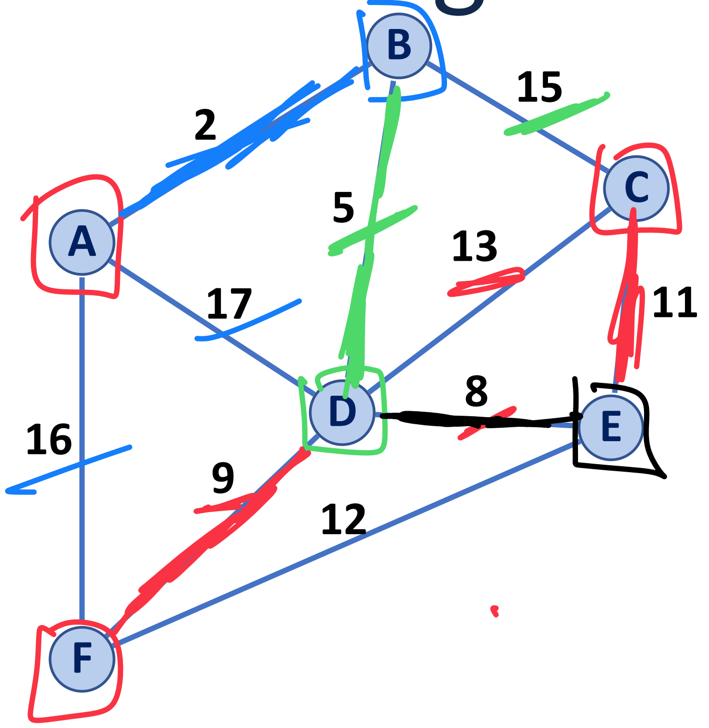
If edge weights can change!

```
1  KruskalMST(G):
2    DisjointSets forest
3    foreach (Vertex v : G.vertices()):
4      forest.makeSet(v)
5
6    PriorityQueue Q    // min edge weight
7    Q.buildFromGraph(G.edges())
8
9    Graph T = (V, {})
10
11   while |T.edges()| < n-1:
12     Vertex (u, v) = Q.removeMin()
13     if forest.find(u) != forest.find(v):
14        T.addEdge(u, v)
15        forest.union( forest.find(u),
16                      forest.find(v) )
17
18   return T
19
```

Why is sorted array good?

Sorted array not destroyed and can be useful in other algorithms!

Prim's Algorithm



A	B	C	D	E	F
0, -	~~∞~~ 2, A	~~∞~~ ~~15, B~~ ~~13, D~~ 11, F	~~∞~~ ~~17, A~~ 5, B	~~∞~~ 8, D	~~∞~~ ~~16, A~~ 9, D

```
PrimMST(G, s):
  Input: G, Graph;
         s, vertex in G, starting vertex
  Output: T, a minimum spanning tree (MST) of G

  foreach (Vertex v : G.vertices()):
    d[v] = +inf
    p[v] = NULL
  d[s] = 0

  PriorityQueue Q   // min distance, defined by d[v]
  Q.buildHeap(G.vertices())
  Graph T           // "labeled set"

  repeat n times:
    Vertex m = Q.removeMin()
    T.add(m)
    foreach (Vertex v : neighbors of m not in T):
      if cost(v, m) < d[v]:
        d[v] = cost(v, m)
        p[v] = m

  return T
```

Init

← All distances are ∞

starts off removing A

update cost/distance if new edge smaller!

update prev so we know source

Prim's Big O

$|V| = n, |E| = m$

7 — 9: O(n)

12—14:

MinHeap: O(n)

Unsorted Array: O(1)

16—22: Complicated!

```
6  PrimMST(G, s):
7    foreach (Vertex v : G.vertices()):
8      d[v] = +inf
9      p[v] = NULL
10   d[s] = 0
11
12   PriorityQueue Q // min distance, defined by d[v]
13   Q.buildHeap(G.vertices())
14   Graph T           // "labeled set"
15
16   repeat n times:
17     Vertex m = Q.removeMin()
18     T.add(m)
19     foreach (Vertex v : neighbors of m not in T):
20       if cost(v, m) < d[v]:
21         d[v] = cost(v, m)
22         p[v] = m
23
```

Depends on choice of **PriorityQueue** (MinHeap vs Unsorted Array)

Depends on choice of **Graph** (Adjacency Matrix vs Adjacency List)

Prim's Algorithm

Sparse Graph: m ~ n

Adj List Heap best

Dense Graph: m ~ n^2

Unsorted Array best

```
PrimMST(G, s):
  foreach (Vertex v : G.vertices()):
    d[v] = +inf
    p[v] = NULL
  d[s] = 0

  PriorityQueue Q // min distance, defined by d[v]
  Q.buildHeap(G.vertices())
  Graph T          // "labeled set"

  repeat n times:
    Vertex m = Q.removeMin()
    T.add(m)
    foreach (Vertex v : neighbors of m not in T):
      if cost(v, m) < d[v]:
        d[v] = cost(v, m)
        p[v] = m
```

	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$
Unsorted Array	$O(n^2)$	$O(n^2)$

MST Algorithm Runtime:

Kruskal's Algorithm:
$O(n + m \log (n))$

Prim's Algorithm:
$O(n \log(n) + m \log (n))$

Sparse Graph: $m \sim n$

Both are $n \log n$

Dense Graph: $m \sim n^2$

Both are $n^2 \log n$

Dijkstra's Algorithm (SSSP)

November 5th

O(m + n log n)

What is the running time of Dijkstra's Algorithm? The same as Prim's!

6-9: O(n)

11-12: O(n)

15: repeat below n x

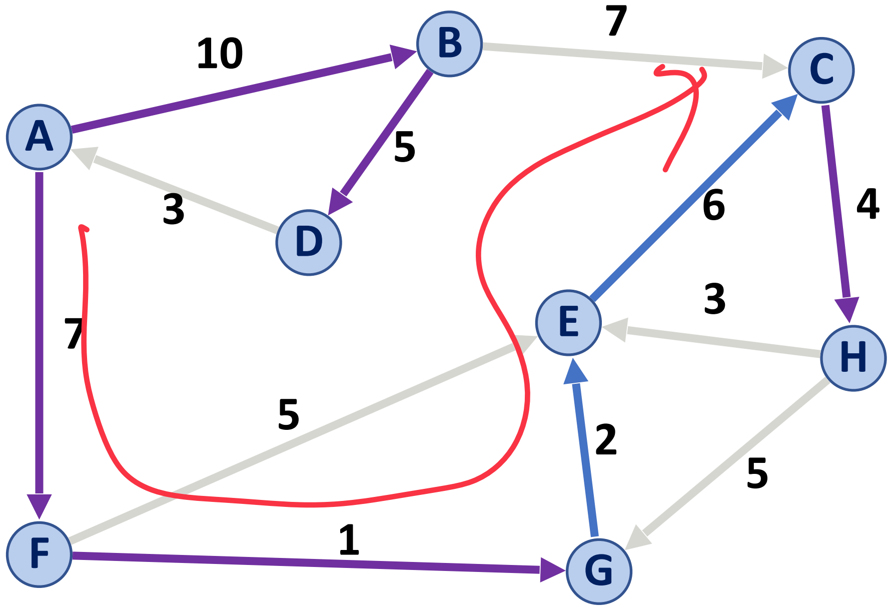
16-22: O(log n)

[w/ Fib Heap O(1) updates]

```
   DijkstraSSSP(G, s):
6    foreach (Vertex v : G):
7      d[v] = +inf
8      p[v] = NULL
9    d[s] = 0
10
11   PriorityQueue Q // min distance, defined by d[v]
12   Q.buildHeap(G.vertices())
13   Graph T          // "labeled set"
14
15   repeat n times:
16     Vertex u = Q.removeMin()
17     T.add(u)
18     foreach (Vertex v : neighbors of u not in T):
19       if cost(u, v) + d[u] < d[v]:
20         d[v] = cost(u, v) + d[u]
21         p[v] = m
22
23   return T
```

edge cost plus prev costs

Dijkstra's Algorithm (SSSP)



This is not MST!

```
DijkstraSSSP(G, s):
  foreach (Vertex v : G.vertices()):
    d[v] = +inf
    p[v] = NULL
  d[s] = 0

  PriorityQueue Q // min distance, defined by d[v]
  Q.buildHeap(G.vertices())
  Graph T          // "labeled set"

  repeat n times:
    Vertex u = Q.removeMin()
    T.add(u)
    foreach (Vertex v : neighbors of u not in T):
      if cost(u, v) + d[u] < d[v]:
        d[v] = cost(u, v) + d[u]
        p[v] = u
```

A	B	C	D	E	F	G	H
--	A	E	B	G	A	F	C
0	10	16	15	10	7	8	20

This solves shortest path

Dijkstra's Algorithm (SSSP)

Dijkstras Algorithm works only on non-negative weights

Optimal implementation:

Fibonacci Heap

If dense, unsorted list ties

Optimal runtime:

Sparse: $O(m + n \log n)$

Dense: $O(n^2)$

```
DijkstraSSSP(G, s):
  foreach (Vertex v : G):
    d[v] = +inf
    p[v] = NULL
  d[s] = 0

  PriorityQueue Q // min distance, defined by d[v]
  Q.buildHeap(G.vertices())
  Graph T          // "labeled set"

  repeat n times:
    Vertex u = Q.removeMin()
    T.add(u)
    foreach (Vertex v : neighbors of u not in T):
      if cost(u, v) + d[u] < d[v]:
        d[v] = cost(u, v) + d[u]
        p[v] = m

  return T
```

Floyd-Warshall Algorithm

Running time? O(n^3)

Easy to code / multi-threadable

Can handle negative weights!

```
FloydWarshall(G):
  Let d be a adj. matrix initialized to +inf
  foreach (Vertex v : G):
    d[v][v] = 0
  foreach (Edge (u, v) : G):
    d[u][v] = cost(u, v)

  foreach (Vertex u : G):
    foreach (Vertex v : G):
      foreach (Vertex w : G):
        if d[u, v] > d[u, w] + d[w, v]:
          d[u, v] = d[u, w] + d[w, v]
```

Final thoughts on Graphs

Graphs have a large space of **possible coding questions**

You've seen graph questions on other exams:

- Make sure you can use graphs to find all neighbors
- Make sure you can use graphs to solve path questions

Consider how these fundamental skills can be challenged

- What if I had labels on nodes and I need to find specific ones?
- What if I need to label nodes or edges with specific properties?
- Can I handle weights? Directions?

Probability in CS

Fundamentals of Probability

November 10th

Imagine you roll a pair of six-sided dice. What is the expected value?

A **random variable** is a function from events to numeric values.

↳ What is the expected dice roll value

"I roll two dice"

$E[\text{two dice rolls}] = 2E[1]$

The **expectation** of a (discrete) random variable is:

Prob of event · Value of event

two dice $\frac{1}{36}(1+1) + \frac{2}{36}(1+2) + \ldots\ldots$

$$E[X] = \sum_{x \in \Omega} Pr\{X = x\} \cdot x$$

Prob of one dice: $\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + 3/6 + 4/6 + 5/6 + 6/6$

$= 3.5$

$E[\text{one dice roll}]$

Probabilistic Data Structures

Sometimes a data structure can be **too ordered / too structured**

Randomized data structures rely on **expected** performance

Randomized data structures 'cheat' tradeoffs!

Randomized Algorithms

A **randomized algorithm** is one which uses a source of randomness somewhere in its implementation.

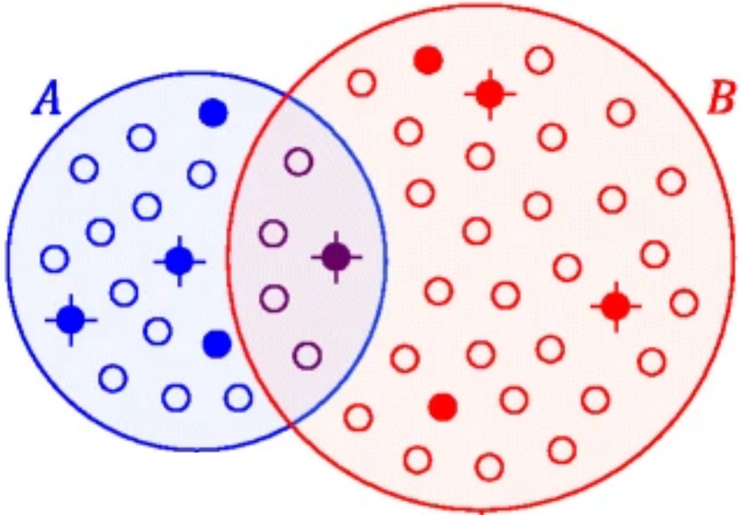
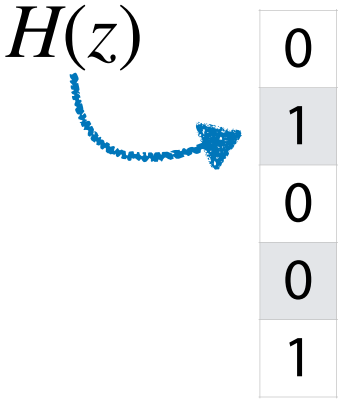
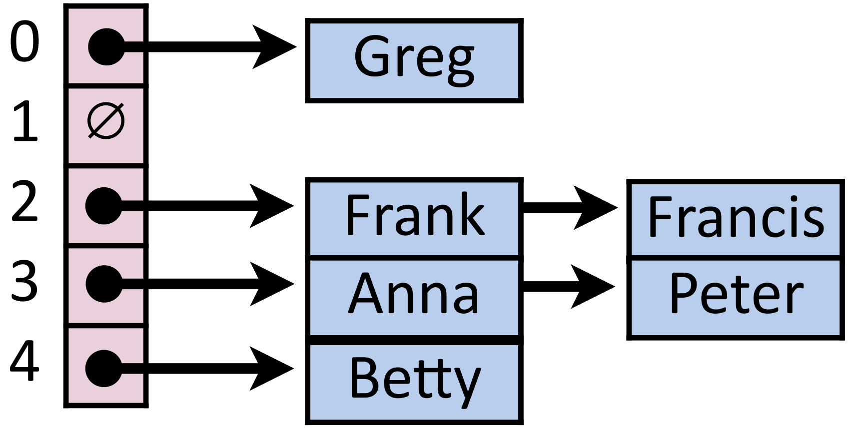


Figure from Ondov et al 2016





$H(x)$	0	2	1	0	0	4	0	2	0	6
$H(y)$	1	0	2	3	1	0	3	4	0	1
$H(z)$	2	1	0	2	0	1	0	0	7	2

A Hash Table based Dictionary

User Code (is a map):

```
Dictionary<KeyType, ValueType> d;
d[k] = v;
```

A **Hash Table** consists of three things:

1. A hash function — Assigns numeric (positive int) address to any key

   Key -> Hash Value (Address)

2. A data storage structure — Array — very good at lookup given **index**

   Hash Value (Address) is an index!

3. A method of addressing *hash collisions*

Two different keys, same hash value

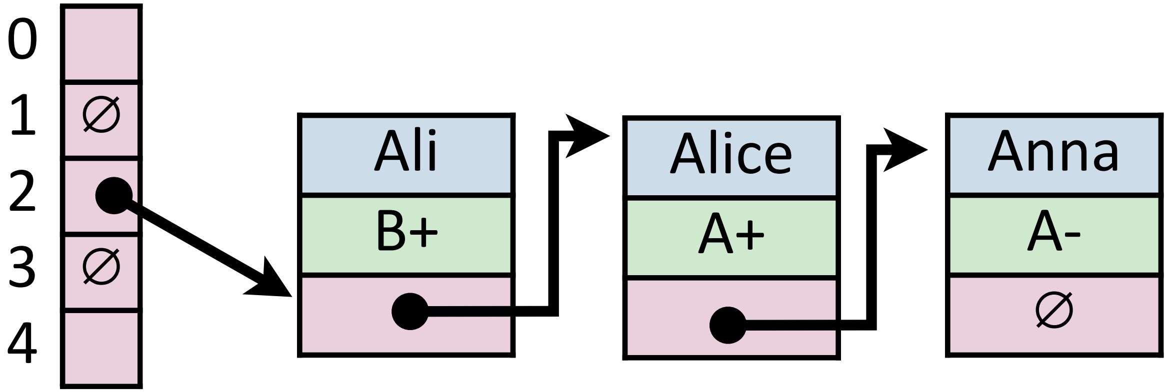
Open vs Closed Hashing

Addressing hash collisions depends on your storage structure.

- **Open Hashing:** store *k,v* pairs externally

Such as a linked list

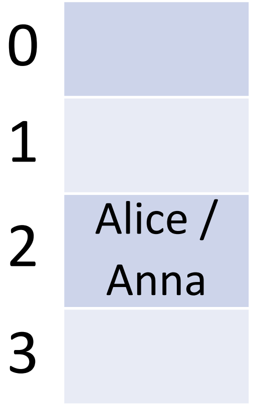
Resolve collisions by adding to list



- **Closed Hashing:** store *k,v* pairs in the hash table

Everything stored in one list

How to store collisions? Unclear!



Simple Uniform Hashing Assumption

Given table of size m, a simple uniform hash, h, implies

$$\forall k_1, k_2 \in U \text{ where } k_1 \neq k_2 ,\ Pr(h[k_1] = h[k_2]) = \frac{1}{m}$$

Uniform: All keys equally likely to hash to any position

$$Pr(h[k_1]) = \frac{1}{m}$$

Independent: All key's hash values are independent of other keys

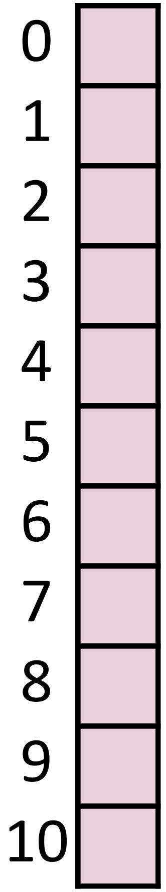
Separate Chaining Under SUHA

Under SUHA, a hash table of size *m* and *n* elements: $\alpha = n/m$

Find runs in: O(1+α).

Insert runs in: O(1).

Remove runs in: O(1+α).



(Example of closed hashing)

Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 } **|S| = n**

h(k, i) = (k + i) % 7 **|Array| = m**

0	22
1	8
2	16
3	29
4	4
5	11
6	13

`_find(29)`

1) Hash the input key [h(29)=1]

2) Look at hash value (address) position

   If present, return (k, v)

   If not look at **next available space**

Stop when:

1) We find the object we are looking for
2) We have searched every position in the array
3) We find a blank space

Running Times (Expectation under SUHA)

Open Hashing: $0 \leq \alpha \leq \infty$ (Length of chain)

insert: ___1___.

find/ remove: ___$1+\alpha$___.

Closed Hashing: $0 \leq \alpha < 1$ (fraction full)

insert: ___$\frac{1}{1-\alpha}$___.

find/ remove: ___$\frac{1}{1-\alpha}$___.

Observe:

* **- As α increases:**

OH: $\alpha \to \infty$, runtime $\to \infty$

CH: $\alpha \to 1$, runtime $\to \infty$

* **- If α is constant:**

OH is constant
CH is constant
] $O(1)^*$

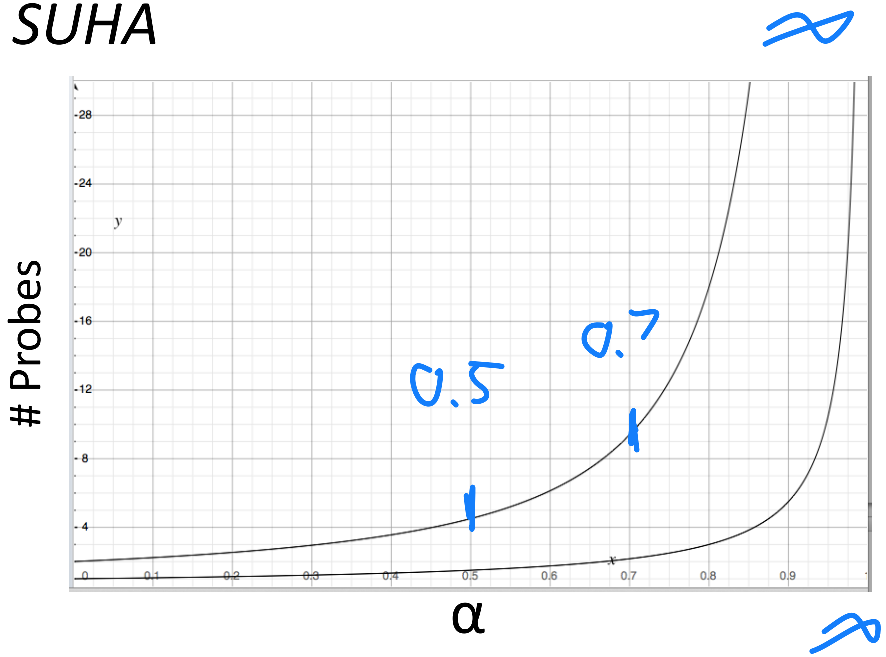
Running Times *(Don't memorize these equations, no need.)*

The expected number of probes for find(key) under SUHA

Linear Probing:

- Successful: **½(1 + 1/(1-α))**
- Unsuccessful: **½(1 + 1/(1-α))²**

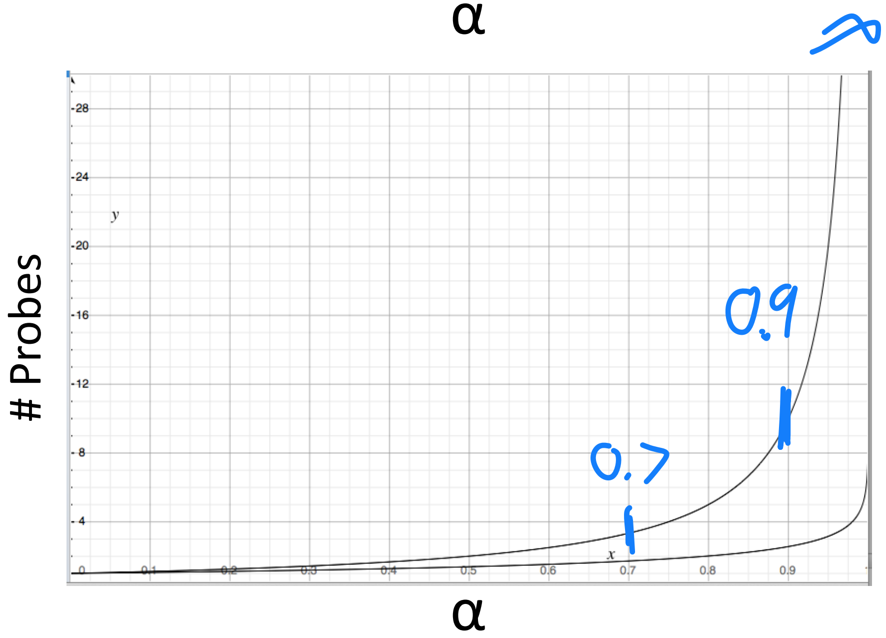
runtime



Double Hashing:

- Successful: **1/α * ln(1/(1-α))**
- Unsuccessful: **1/(1-α)**

runtime



When do we resize?

Linear ~0.7-0.8

Double ~0.7-0.9

Running Times (Tradeoff Highlights)

	Hash Table	AVL	Linked List
Find	Expectation*: O(1)*** Worst Case: O(n)	O(log n)	O(n)
Insert	Expectation*: O(1)*** Worst Case: O(n) Separate Chaining: O(1)	O(log n)	O(1)
Storage Space	O(n)	O(n)	O(n)

Bloom Filter

$H = \{h_1, h_2, \ldots, h_k\}$

A probabilistic data structure storing a set of values

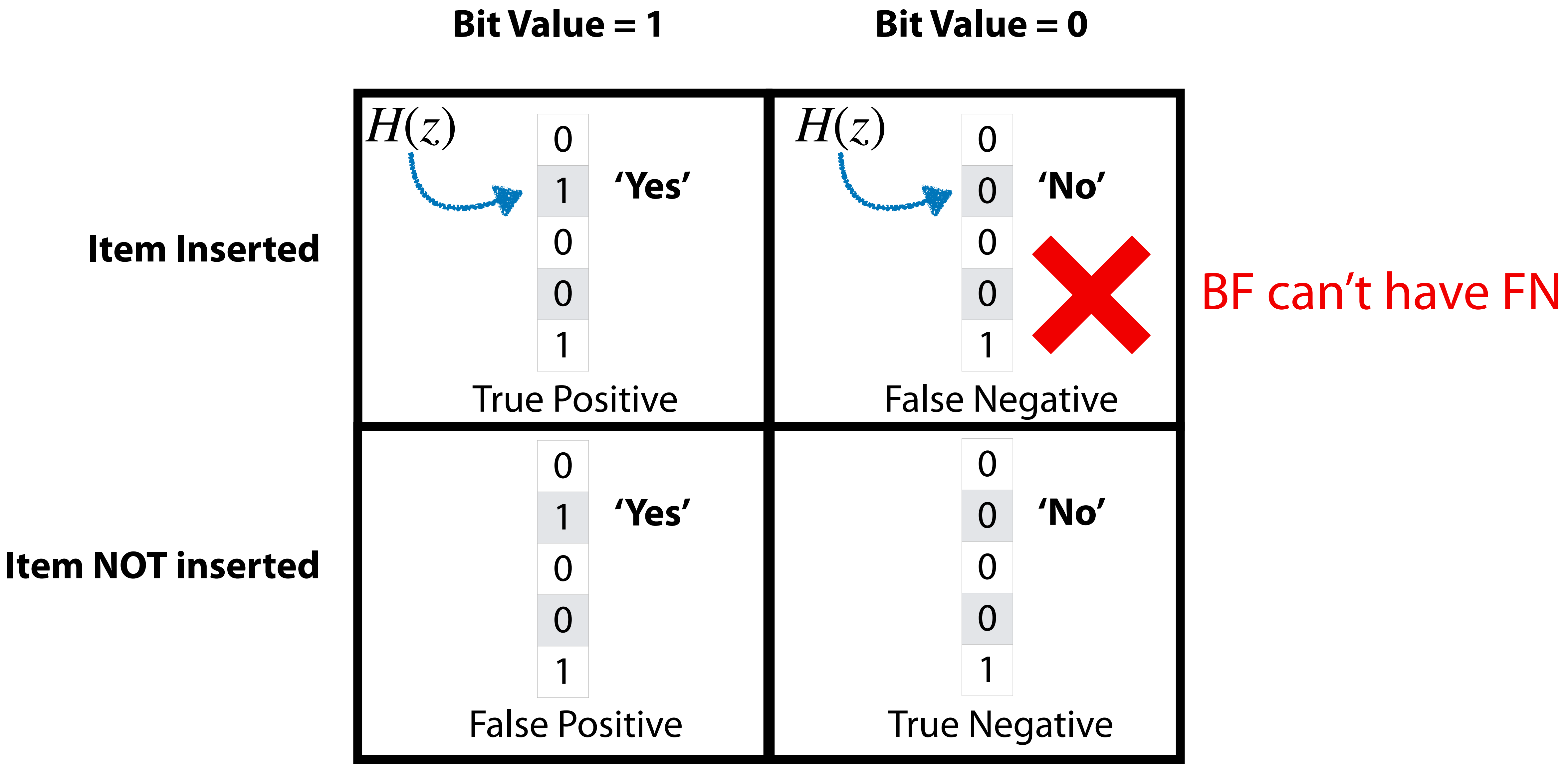
Built from a bit vector of length m and k hash functions

Insert / Find runs in: $O(k)$ / $O(1)$

Delete is not possible (yet)!

0
0
1
0
0
1
0
1
0
0

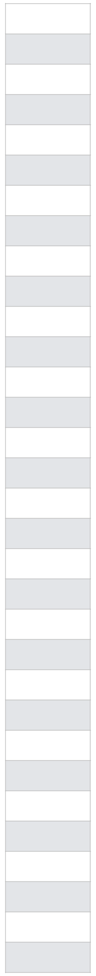
Probabilistic Accuracy in a Bloom Filter



Bloom Filters

A probabilistic data structure storing a set of values

$h_{\{1,2,3,\ldots,k\}}$



Has three key properties:

- k, number of hash functions
- n, expected number of insertions
- m, filter size in bits

Expected false positive rate: $$\left(1-\left(1-\frac{1}{m}\right)^{nk}\right)^{k} \approx \left(1-e^{\frac{-nk}{m}}\right)^{k}$$

Optimal accuracy when: $$k^* = \ln 2 \cdot \frac{m}{n}$$

Bloom Filter: Error Rate

Not enough random trials

BF is too saturated

$$\left(1 - e^{\frac{-nk}{m}}\right)^k$$

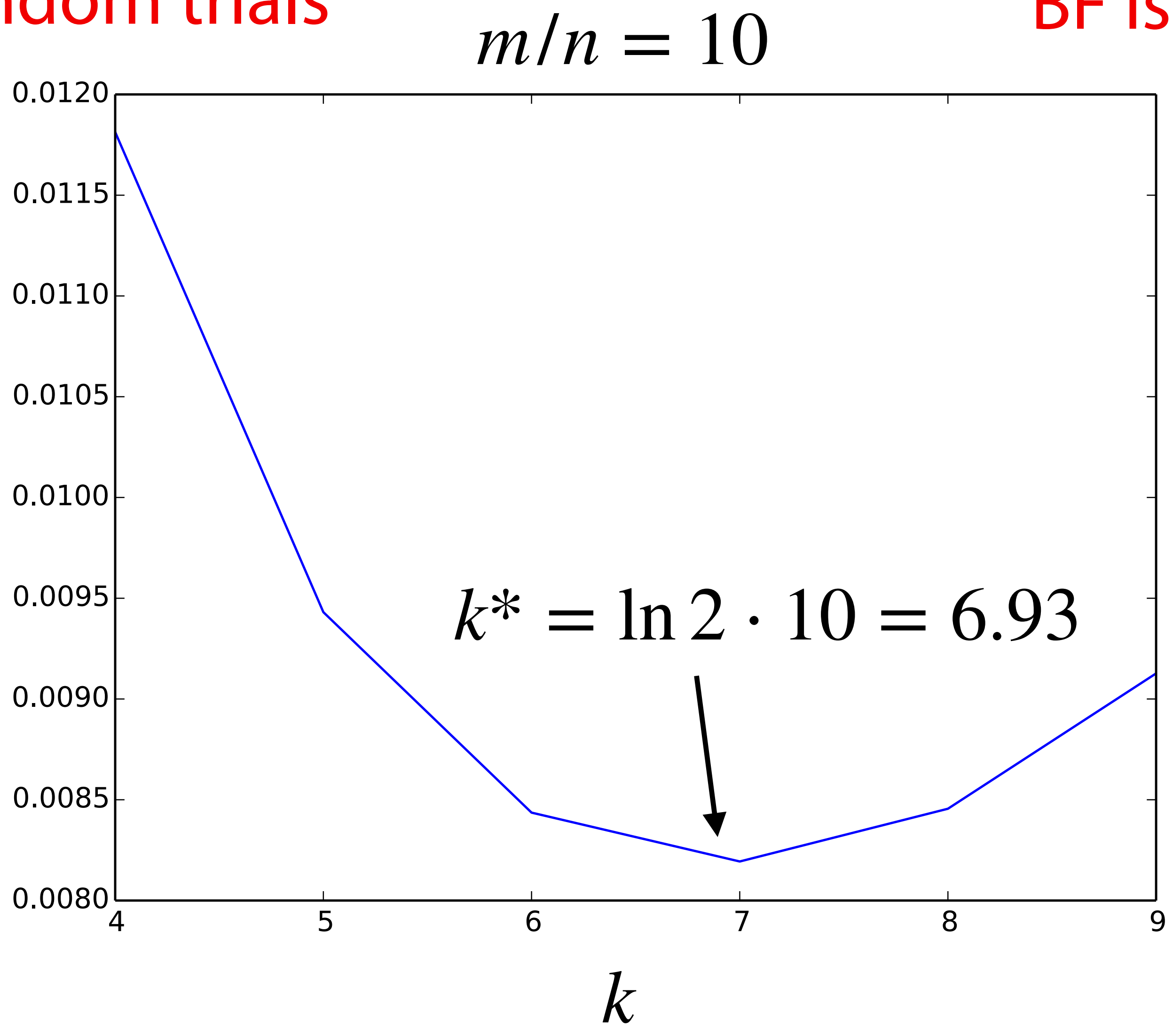


Figure by Ben Langmead

Cardinality Estimation

Let min = 95. Can we estimate N, the cardinality of the set?

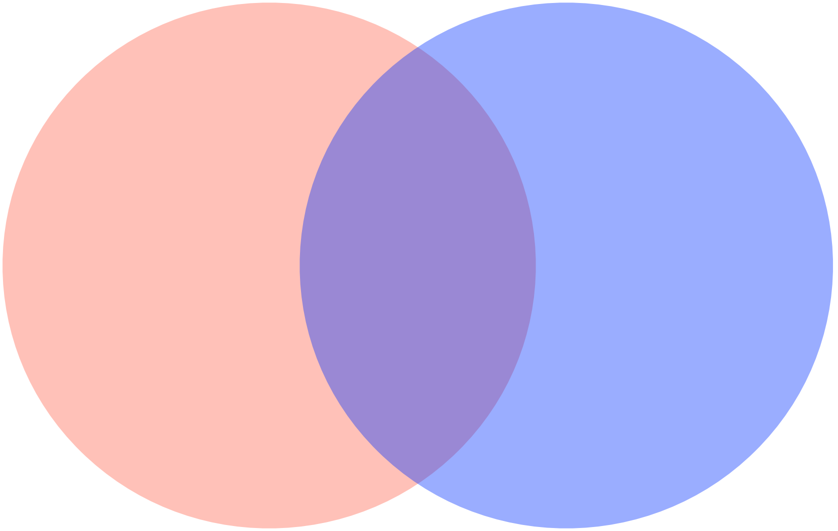
Conceptually: If we scatter N points randomly across the interval, we end up with $N + 1$ partitions, each about $1000/(N+1)$ long

Assuming our first 'partition' is about average:

$$95 \approx 1000/(N+1)$$
$$N + 1 \approx 10.5$$
$$N \approx 9.5$$

Set Similarity Review

To measure **similarity** of A & B, we need both a measure of how similar the sets are but also the total size of both sets.



$$J = \frac{|A \cap B|}{|A \cup B|}$$

J is the ***Jaccard coefficient***

MinHash Sketch

Claim: Under SUHA, set similarity can be estimated by sketch similarity!

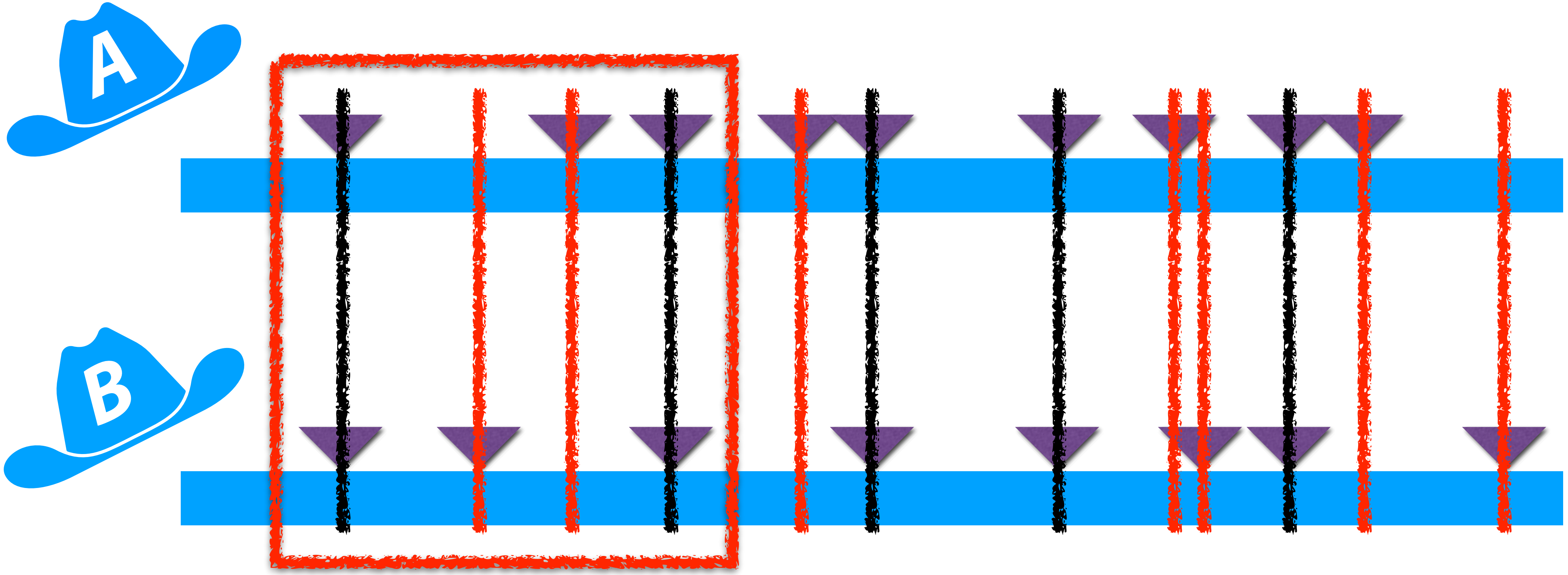
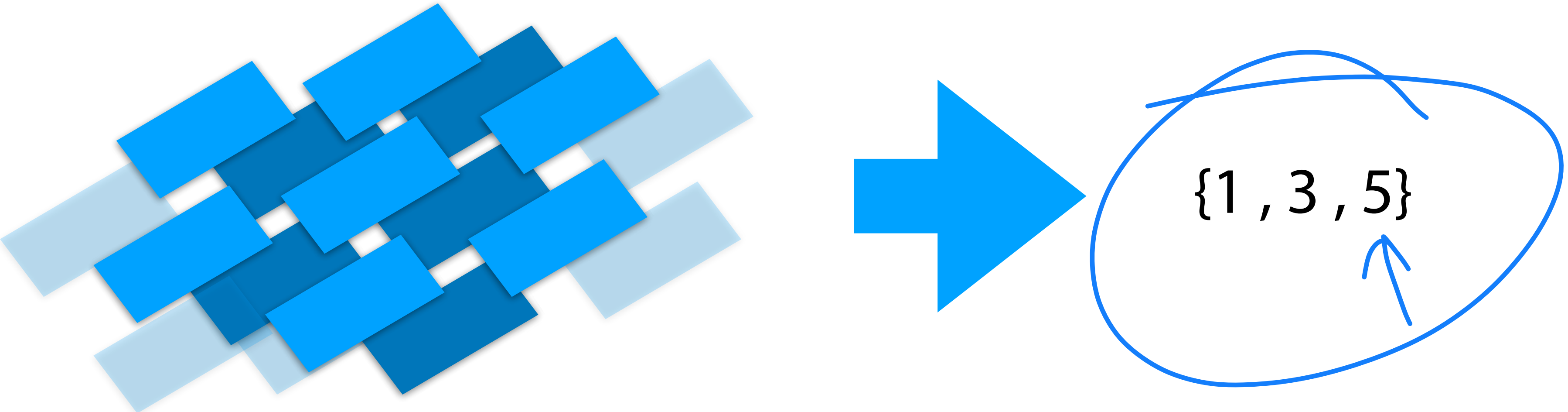


Image inspired by: Ondov B, Starrett G, Sappington A, Kostic A, Koren S, Buck CB, Phillippy AM. **Mash Screen: high-throughput sequence containment estimation for genome discovery**. *Genome Biol* 20, 232 (2019)

MinHash Sketch

We can convert any hashable dataset into a **MinHash sketch**



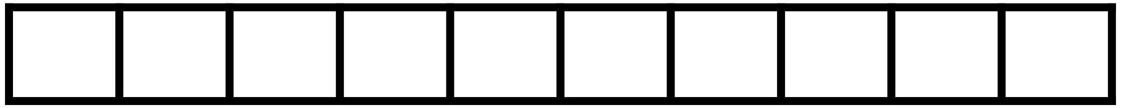
We lose our original dataset, but we can still estimate two things:

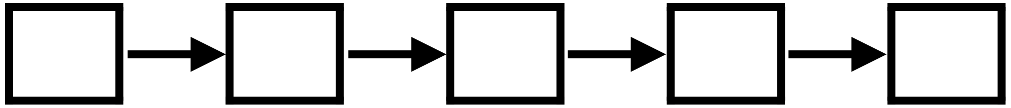
1. Cardinality Estimation (Using the k-th minimum hash value)
2. Set Similarity (Using all k-min hash values)

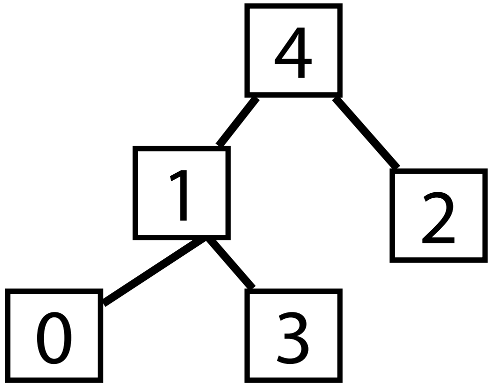
Questions?

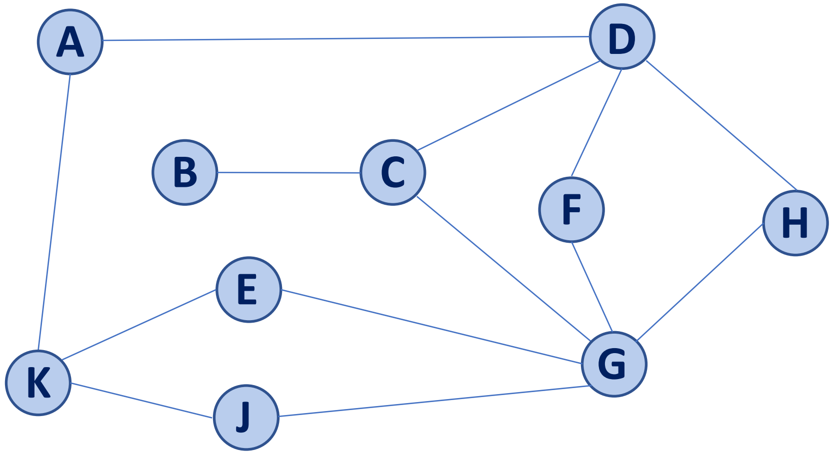
CS 225 — Course Goals

Understand foundational data structures and algorithms







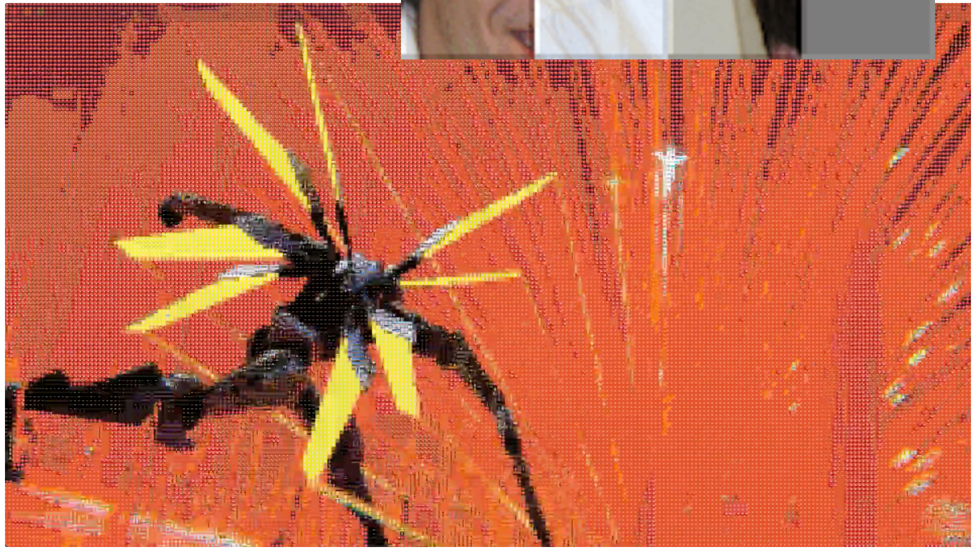


CS 225 — Course Goals

Justify appropriate algorithms for complex problems

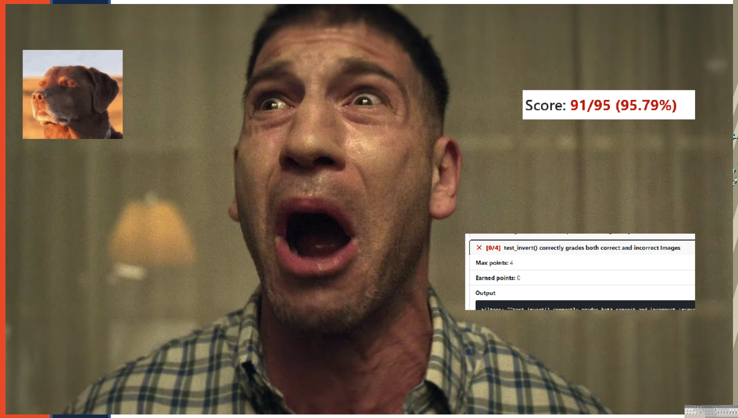
Decompose problem into supporting data structures

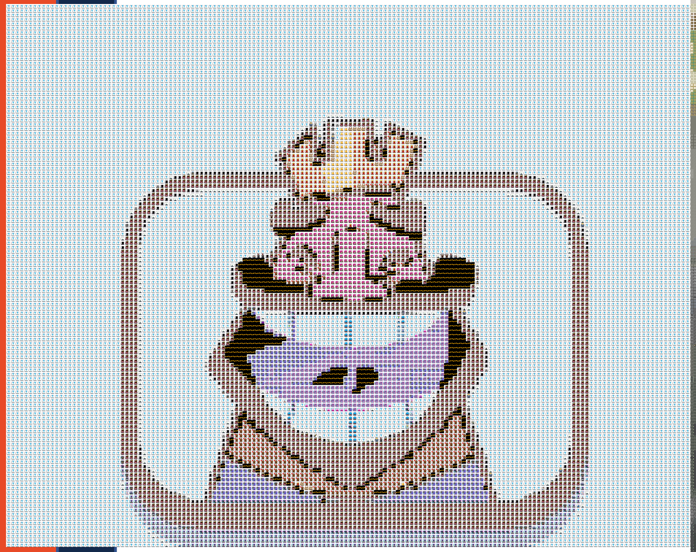
Analyze efficiency of implementation choices

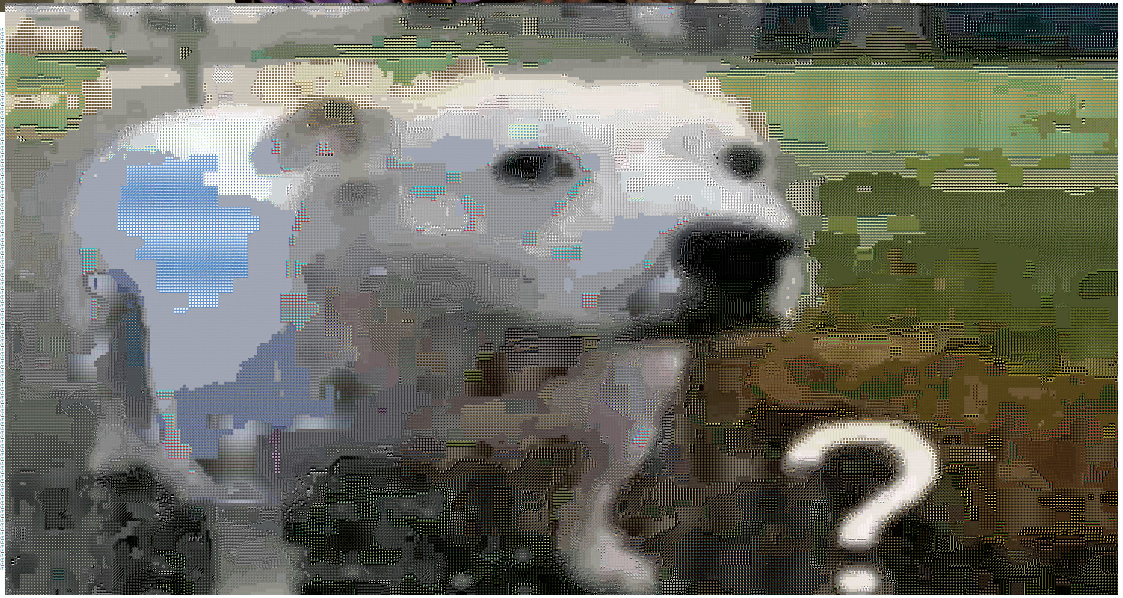


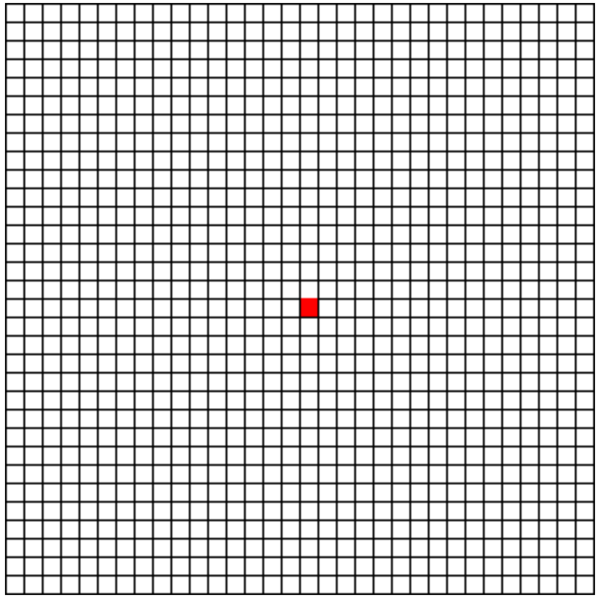
CS 225 — Course Goals

Implement intermediate difficulty problems in C++









CS 225 — Course Goals

Understand foundational data structures and algorithms

Justify appropriate algorithms for complex problems

Implement intermediate difficulty problems in C++

Improve your foundation of CS theory

Good luck on your finals!