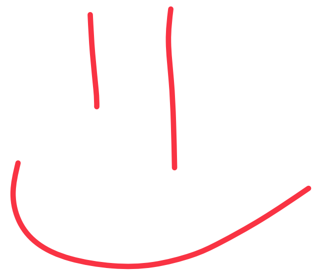
Data Structures and Algorithms

Hashing 3



CS 225
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November 17, 2025

UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN
Department of Computer Science

Announcements

MP_Puzzle releases today

Next week is Fall break (does not count as week for assignments)

Exam 5 is immediately after fall break!

Exam Retake is 12/7 — 12/9

Exam 0 - 4 are allowed retakes
Pick One
Grade is average of both (this can hurt you)

There are no guarantees about overrides on the retake exam!

Learning Objectives

Review hash table implementations

Improve our closed hash implementation

Determine when and how to resize a hash table

Justify when to use different index approaches

Simple Uniform Hashing Assumption

Given table of size m, a simple uniform hash, h, implies

$$\forall k_1, k_2 \in U \text{ where } k_1 \neq k_2,\ Pr(h[k_1] = h[k_2]) = \frac{1}{m}$$

Uniform: All keys equally likely to hash to any position

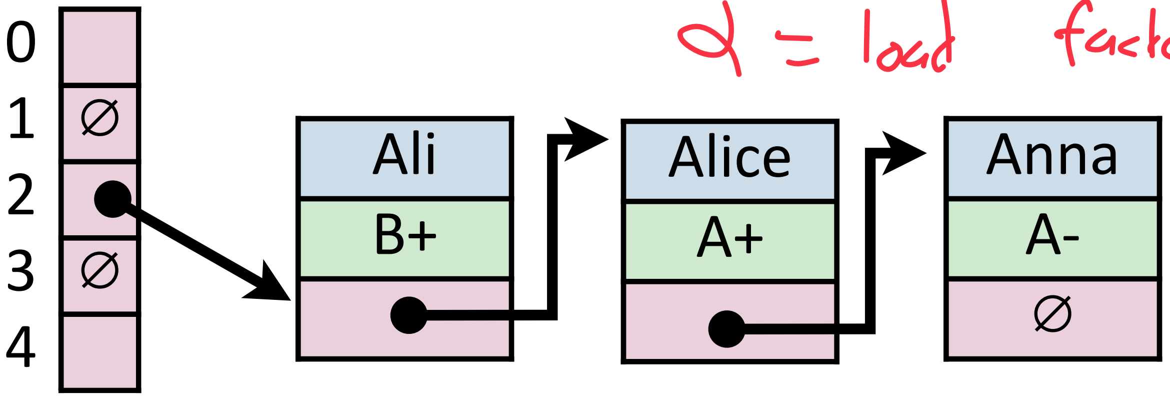
$$Pr(h[k_1]) = \frac{1}{m}$$

Independent: All key's hash values are independent of other keys

Open vs Closed Hashing

Addressing hash collisions depends on your storage structure.

- **Open Hashing:** store *k,v* pairs externally



- **Closed Hashing:** store *k,v* pairs in the hash table

0	Anna, A-
1	
2	Ali, B+
3	Alice, A+

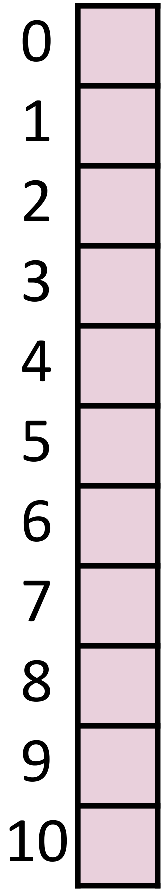
Separate Chaining Under SUHA

Under SUHA, a hash table of size *m* and *n* elements:

Find runs in: O(1+α).

Insert runs in: O(1).

Remove runs in: O(1+α).



(Example of closed hashing)

Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 }

h(k) = k % 7

22%7 = 1

|S| = n

|Array| = m

0	22
1	8
2	16
3	29
4	4
5	11
6	13

h(k, i) = (k + i) % 7

Try h(k) = (k + 0) % 7, if full…

Try h(k) = (k + 1) % 7, if full…

Try h(k) = (k + 2) % 7, if full…

Try …

(Example of closed hashing)

Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 }

|S| = n

h(k, i) = (k + i) % 7

|Array| = m

0	22
1	8
2	16
3	29
4	4
5	11
6	13

`_find(29)`

1) Hash the input key [h(29)=1]

O(1) calculation

2) Look at hash value (address) position

If present, return (k, v)

If not look at **next available space**

Stop when:

1) We find the object we are looking for
2) We have searched every position in the array
3) We find a blank space

Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 }

h(k, i) = (k + i) % 7

|S| = n

|Array| = m

0	22
1	8
2	16
3	29
4	4
5	11
6	13

`_remove(16)`

1) Hash the input key [h(16)=2]
2) Find the actual location (if it exists)
3) Remove the (k,v) at hash value (address)

Don't resize the array! Tombstone!

A Problem w/ Linear Probing

Primary Clustering: "Rich get richer"

Many more collisions than we want!

0	
1	1_1
2	1_2
3	3_1
4	1_3
5	3_2
6	
7	
8	
9	

Description:

Collisions create long runs of filled-in indices

Should have a **1/m** chance to hash anywhere

Instead have a **(size of cluster) / m** chance to hash at end

Remedy: Pick better next available

A Problem w/ Linear Probing

Primary Clustering: "Rich get richer"

0	
1	**1_1**
2	**1_2**
3	**3_1**
4	**1_3**
5	**3_2**
6	
7	
8	
9	

Description:

Collisions create long runs of filled-in indices

Should have a **1/m** chance to hash anywhere

Instead have a **(size of cluster) / m** chance to hash at end

Remedy:

Pick a better "next available" position!

(Example of closed hashing)

Collision Handling: Quadratic Probing

S = { 16, 8, 4, 13, 29, 12, 22 }

|S| = n

h(k) = k % 7

|Array| = m

29 % 7 = 1

1 + 1 * 1

1 + 2 * 2 (4) = 5

0	
1	**8**
2	**16**
3	
4	**4**
5	29
6	**13**

h(k, i) = (k + i*i) % 7

Try h(k) = (k + 0) % 7, if full… +0

Try h(k) = (k + 1*1) % 7, if full… +1

Try h(k) = (k + 2*2) % 7, if full… +4

Try … +9

+16

+25

Can this infinite loop?

↳ If hash table size is prime, no

A Problem w/ Quadratic Probing

Secondary Clustering:

0	0_1
1	0_2
2	
3	
4	0_3
5	
6	
7	
8	
9	0_4

Description:

Individual collisions still yield long chains

Remedy:

Be less consistent (but still deterministic)

(Example of closed hashing)

Collision Handling: Double Hashing

S = { 16, 8, 4, 13, 29, 11, 22 }

|S| = n

|Array| = m

$h_1(k) = k \% 7$

$h_2(k) = 5 - (k \% 5)$

0	22
1	8
2	16
3	29
4	4
5	11
6	13

$h(k, i) = (h_1(k) + i*h_2(k)) \% 7$

Try $h(k) = (k + 0*h_2(k)) \% 7$, if full...

Try $h(k) = (k + 1*h_2(k)) \% 7$, if full...

Try $h(k) = (k + 2*h_2(k)) \% 7$, if full...

Try ...

Running Times

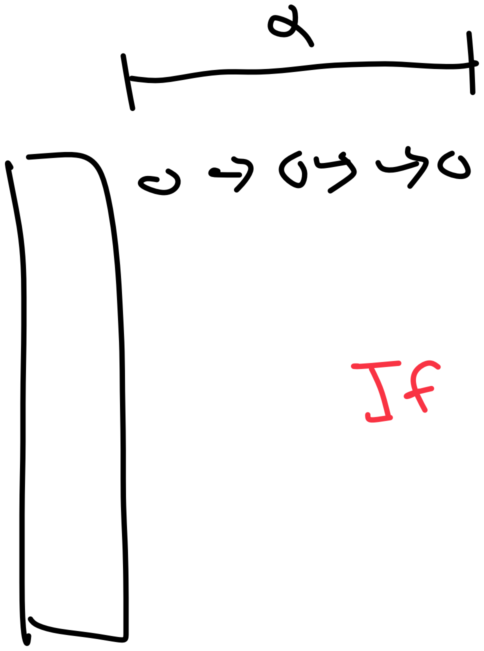
(Expectation under SUHA)

(Understand why we have these rough forms)

Open Hashing:

insert: 1 .

find/ remove: $1 + \alpha$.



If α is fraction < 1

$0 \le \alpha < 1$

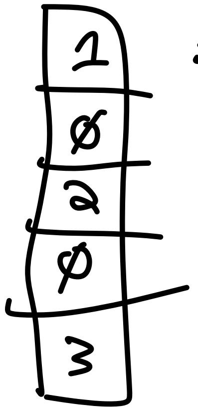
Closed Hashing:

insert: $\frac{1}{1-\alpha}$.

find/ remove: $\frac{1}{1-\alpha}$.

$\alpha = n/m$

(How many positions are filled)



\# of attempts

$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \dots$

one attempt

two attempt

collide once

colliding twice

Running Times (Expectation under SUHA)

Open Hashing: $0 \leq \alpha \leq \infty$

$\frac{n}{m}$ ← can be ∞

insert: 1.

find/ remove: $1 + \alpha$.

Closed Hashing: $0 \leq \alpha < 1$

← $0 \leq n < m$

$\alpha \to 1$

$\alpha = 0.5$

insert: $\frac{1}{1-\alpha}$.

find/ remove: $\frac{1}{1-\alpha}$.

Observe:

$\alpha \to \infty$

- As α increases:

runtime increases to ∞

- If α is constant:

↳ runtime is a constant

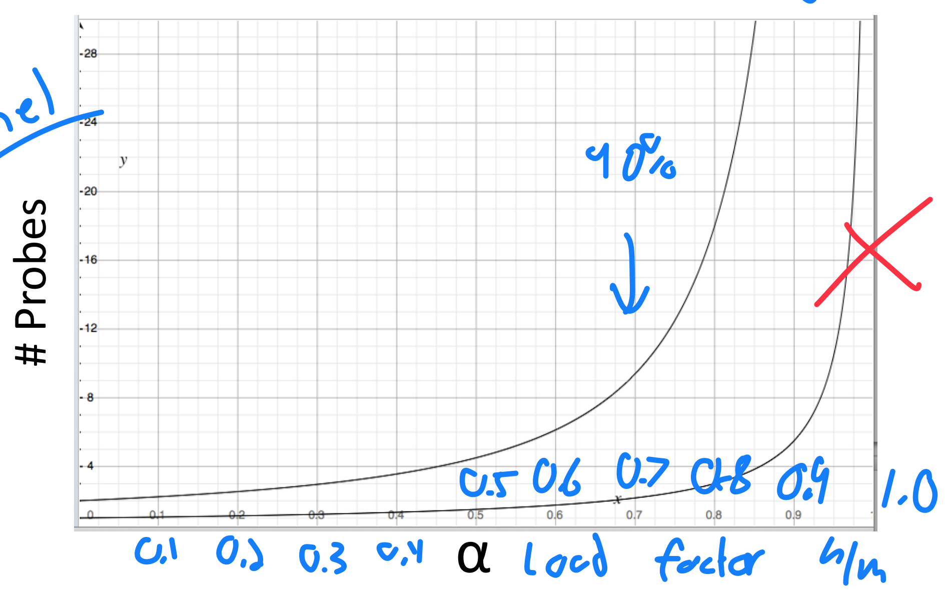
:)

Running Times *(Don't memorize these equations, no need.)*

The expected number of probes for find(key) under SUHA

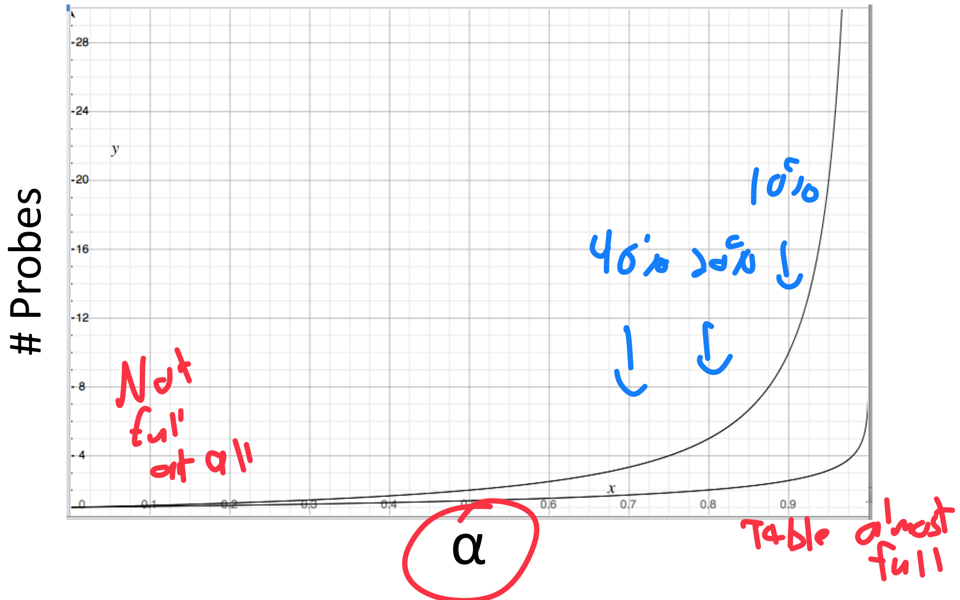
Linear Probing:

- Successful: **½(1 + 1/(1-α))**
- Unsuccessful: **½(1 + 1/(1-α))²**



Double Hashing:

- Successful: **1/α * ln(1/(1-α))**
- Unsuccessful: **1/(1-α)**



When do we resize?

LP: 0.5 - 0.7

DH: 0.7 - 0.9

Resizing a hash table

How do you resize?

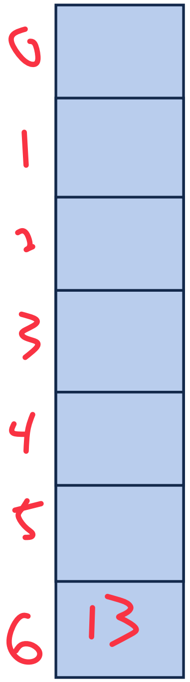
↳ Resize at 90% capacity

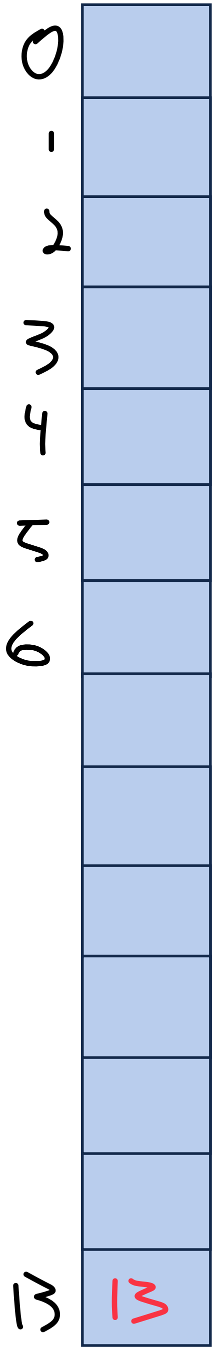
$h(k) = k \% 7$ →

1) Double size of array and make new hash

2) ~~Copy over all items~~

2) No! Rehash all items





$h(k) = k \% 14$

Claim: Hash table insert is $O(1)$ **

* This is assuming SUHA (expectation)

* This is pseudo-amortized
↳ I resize at 0.9 capacity

* When I double, I 'double' to a new prime #

Tradeoffs! I can resize earlier if speed matters

Cost: Insert gets 'slower'

Which collision resolution strategy is better?

- Big Records: Open hash (pass by ref)
- Structure Speed: Closed hashing

What structure do hash tables implement? Dictionary

What constraint exists on hashing that doesn't exist with BSTs?

↳ amortized, probabilistic, SUHA

Why talk about BSTs at all? Range find, approximate find

std::map in C++

```
T& map<K, V>::operator[]
pair<iterator, bool> map<K, V>::insert()
iterator map<K, V>::erase()


iterator map<K, V>::lower_bound( const K & );
iterator map<K, V>::upper_bound( const K & );
```

std::unordered_map in C++

```
T& unordered_map<K, V>::operator[]
pair<iterator, bool> unordered_map<K, V>::insert()
iterator unordered_map<K, V>::erase()
```

~~`iterator map<K, V>::lower_bound( const K & );`~~

~~`iterator map<K, V>::upper_bound( const K & );`~~

```
float unordered_map<K, V>::load_factor();
void unordered_map<K, V>::max_load_factor(float m);
```

Running Times

Added after class for study purposes

	Hash Table	AVL	Linked List
Find	Expectation*: O(1) Worst Case: O(n)		
Insert	Expectation*: O(1) Worst Case: O(n) O(1) for separate chain!		
Storage Space			

Next Class: Probabilistic Accuracy in Data Structures

1. Assume input data is random to estimate average-case performance

2. Use randomness inside algorithm to estimate expected running time

3. Use randomness inside algorithm to approximate solution in fixed time

Probabilistic Accuracy: Fermat primality test

Pick a random a in the range $[2, p-2]$

If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$

But… ***sometimes*** if n is composite and $a^{n-1} \equiv 1 \pmod{n}$

Probabilistic Accuracy: Fermat primality test

Let's assume $\alpha = .5$

First trial: $a = a_0$ and prime test returns 'prime!'

Second trial: $a = a_1$ and prime test returns 'prime!'

Third trial: $a = a_2$ and prime test returns 'not prime!'

Is our number prime?

What is our **false positive** probability? Our **false negative** probability?

Probabilistic Accuracy: Fermat primality test

	$a^{p-1} \equiv 1 \pmod{p}$	$a^{p-1} \not\equiv 1 \pmod{p}$
p is prime		
p is not prime		

Probabilistic Accuracy: Fermat primality test

Summary: Randomized algorithms can also have fixed (or bounded) runtimes at the cost of probabilistic accuracy.

Randomness:

Assumptions:

Probabilistic Accuracy: Fermat primality test

Summary: Randomized algorithms can also have fixed (or bounded) runtimes at the cost of probabilistic accuracy.

Randomness: The choice of α.

We can even pick more than one α if we want!

Assumptions: Only that random numbers are actually random

While strictly not true, generally an acceptable assumption in practice

Types of randomized algorithms

A **Las Vegas** algorithm is a randomized algorithm which will always give correct answer if run enough times but has no fixed runtime.

A **Monte Carlo** algorithm is a randomized algorithm which will run a fixed number of iterations and may give the correct answer.

What type of algorithm is Fermat's primality test?

What type of algorithm is randomized quick sort?

Bonus Slides

Hash Table

Worst-Case behavior is bad — but what about randomness?

1) **Fix *h***, our hash, and assume it is good for ***all keys***:

Simple Uniform Hashing Assumption

(Assume our dataset hashes optimally)

2) Create a ***universal hash function family:***

Given a collection of hash functions, pick one randomly

Like **random quicksort** if pick of hash is random, good expectation!

Hash Function (Division Method)

Hash of form: $h(k) = k \,\%\, m$

Pro:

Con:

Hash Function (Mid-Square Method)

Hash of form: $h(k) = (k * k)$ and take b bits from middle ($m = 2^b$)

Hash Function (Mid-Square Method)

Hash of form: $h(k) = (k * k)$ and take b bits from middle $(m = 2^b)$

Hash Function (Multiplication Method)

Hash of form: $h(k) = \lfloor m(kA \,\%\, 1) \rfloor$, $0 \leq A \leq 1$

Pro:

Con:

Hash Function (Universal Hash Family)

Hash of form: $h_{ab}(k) = \left((ak + b)\,\%\,p\right)\%\,m,\ a, b \in Z_p^*, Z_p$

$$\forall k_1 \neq k_2,\ Pr_{a,b}(h_{ab}[k_1] = h_{ab}[k_2]) \leq \frac{1}{m}$$

Pro:

Con: