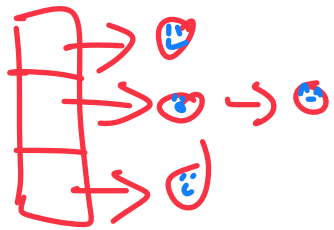
Data Structures and Algorithms

Hashing 2

CS 225

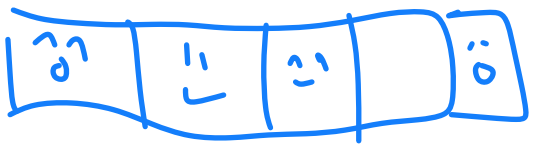
Brad Solomon

November 14, 2025



UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Department of Computer Science



Learning Objectives

O(1)

O(n)
currently

Review fundamentals of hash tables

Introduce closed hashing approaches to hash collisions

Determine when and how to resize a hash table

Justify when to use different index approaches

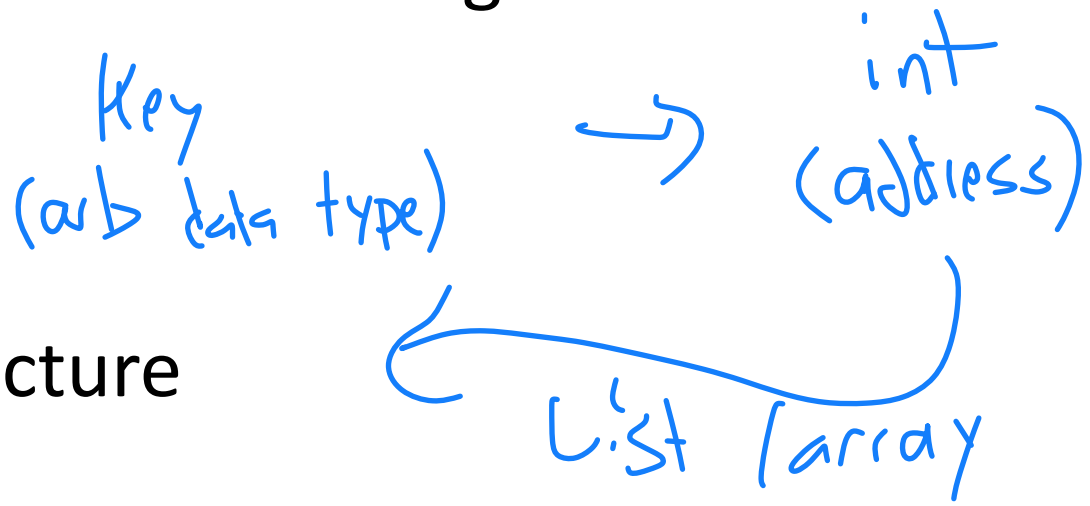
A Hash Table based Dictionary

User Code (is a map):

```
1  Dictionary<KeyType, ValueType> d;
2  d[k] = v;
```

A **Hash Table** consists of three things:

1. A hash function



2. A data storage structure

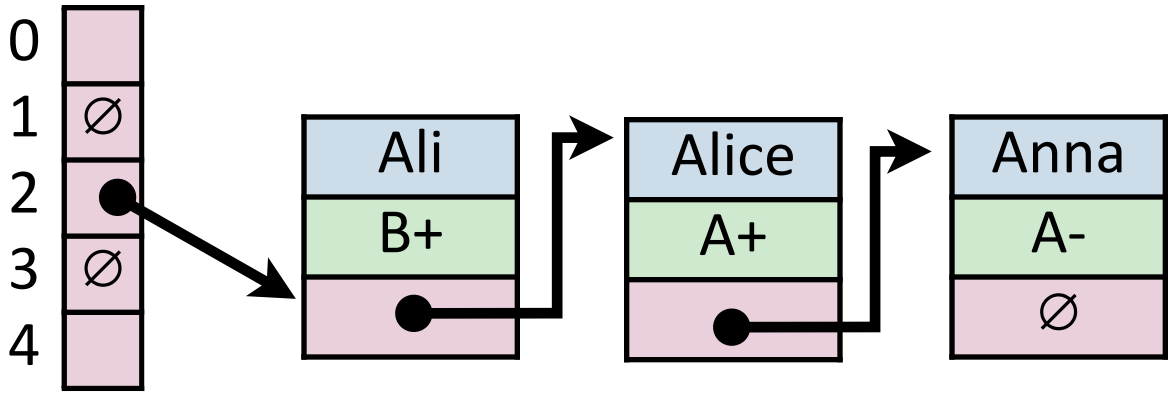
3. **A method of addressing *hash collisions***

$h(K_1) == h(K_2)$

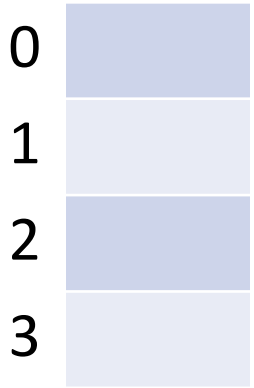
Open vs Closed Hashing

Addressing hash collisions depends on your storage structure.

- **Open Hashing:** store *k,v* pairs externally



- **Closed Hashing:** store *k,v* pairs in the hash table



easy collision handling

Hash Table (Separate Chaining)

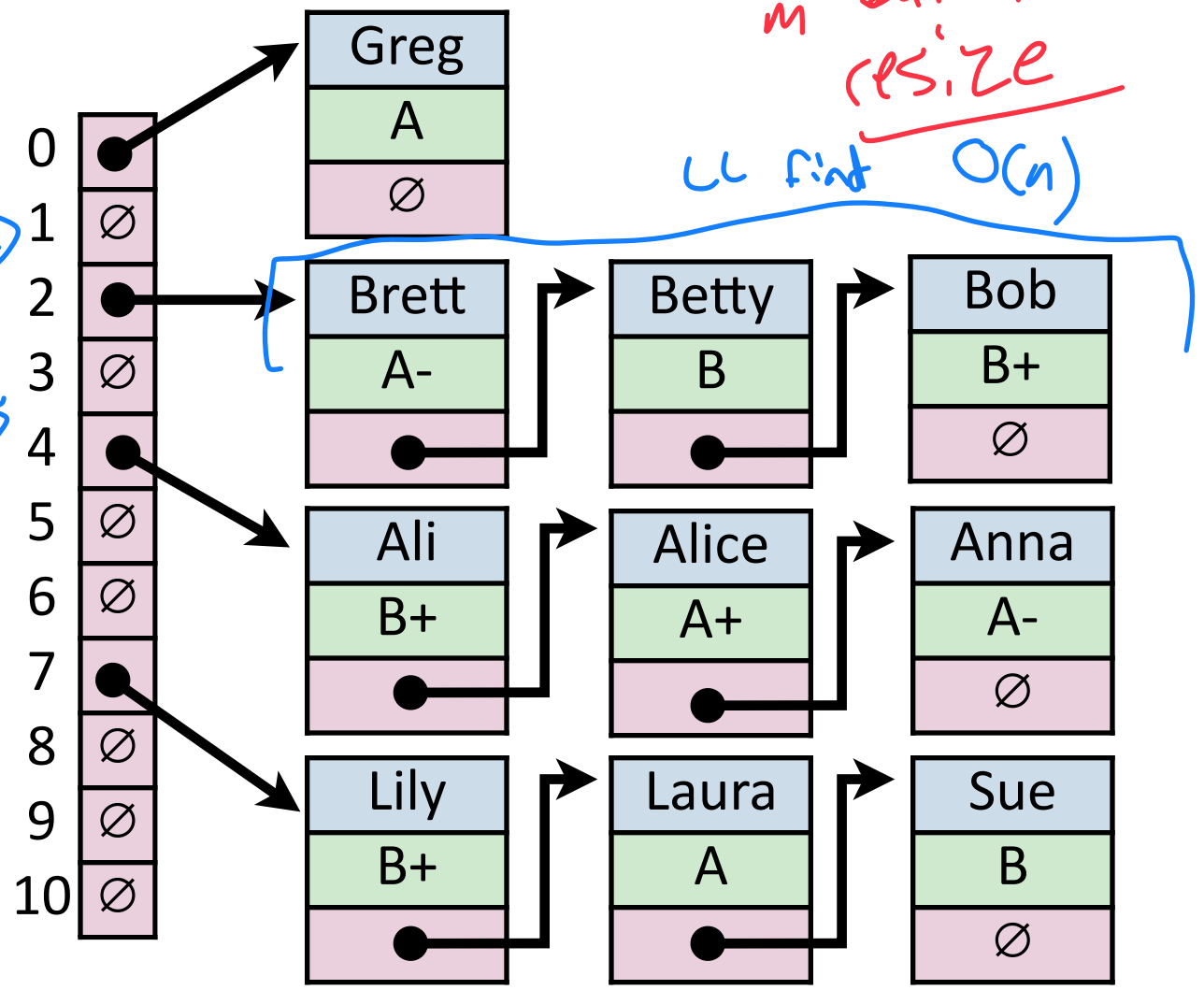
h(k) % m

can change m but this is resize

h(k) → compute address O(1) → Hash addresses not stored

Key	Value	Hash
Bob	B+	2
Anna	A-	4
Alice	A+	4
Betty	B	2
Brett	A-	2
Greg	A	0
Sue	B	7
Ali	B+	4
Laura	A	7
Lily	B+	7

O(1) lookup address



Hash Table (Separate Chaining)

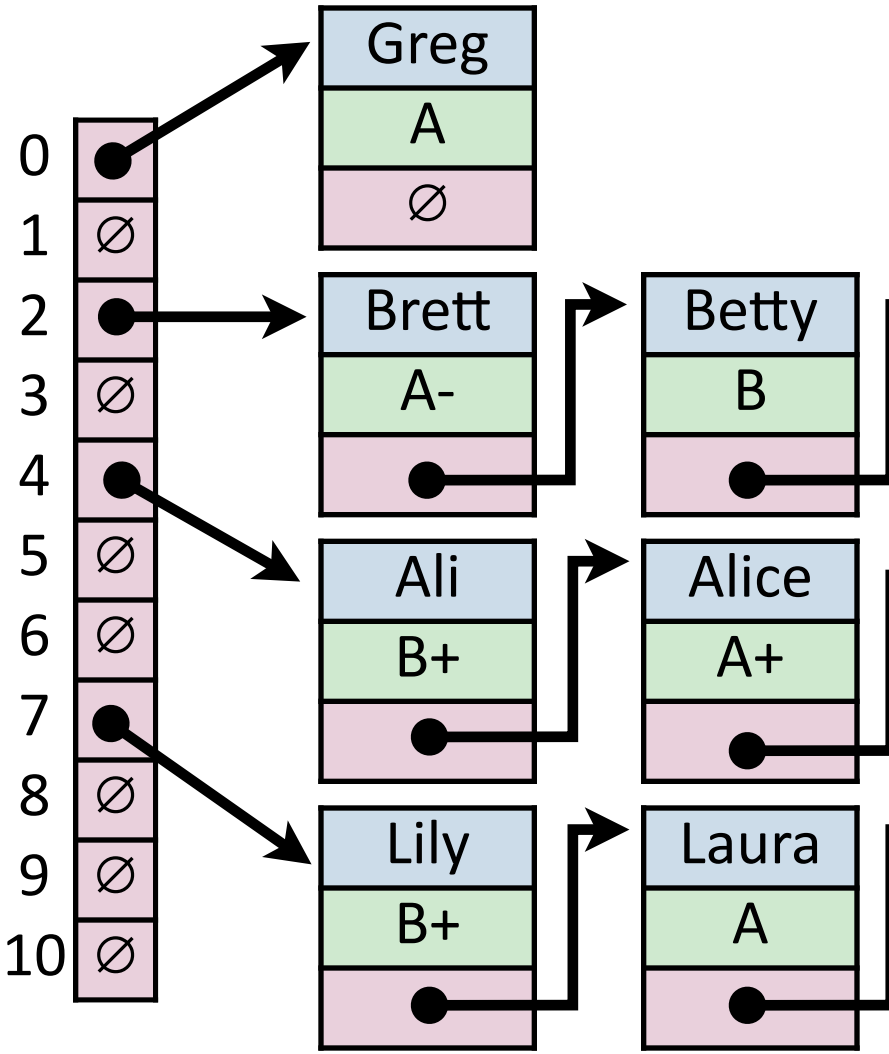
For hash table of size *m* and *n* elements:

Find runs in: O(n)

Insert runs in: O(1)

Remove runs in: O(n)

we must be able to claim O(n) ⇒ O(1)**



Hash Table

Expected Performance

Worst-Case behavior is bad — but what about randomness?

1) **Fix *h***, our hash, and assume it is good for ***all keys***: on our data

simple uniform hashing assumption (weak argument)

and yet

2) Create a ***universal hash function family:*** on our choice of a hash function

↳ Randomness is on choice of hash function

(Beyond 225)

Hash Table

Worst-Case behavior is bad — but what about randomness?

1) **Fix *h***, our hash, and assume it is good for ***all keys***:

Simple Uniform Hashing Assumption

(Assume our dataset hashes optimally)

2) Create a ***universal hash function family:***

Given a collection of hash functions, pick one randomly

Like random quicksort if pick of hash is random, good expectation!

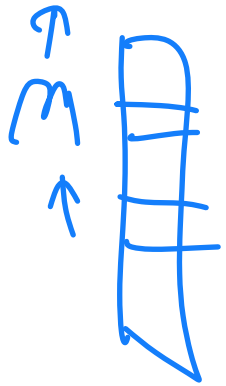
Simple Uniform Hashing Assumption

225 - level

Given table of size m, a simple uniform hash, h, implies

$$\forall k_1, k_2 \in U \text{ where } k_1 \neq k_2,\ Pr(h[k_1] = h[k_2]) = \frac{1}{m}$$

Uniform: All positions / addresses / values are equally likely



Independent: $h(k_1)$ & $h(k_2)$ hash independently

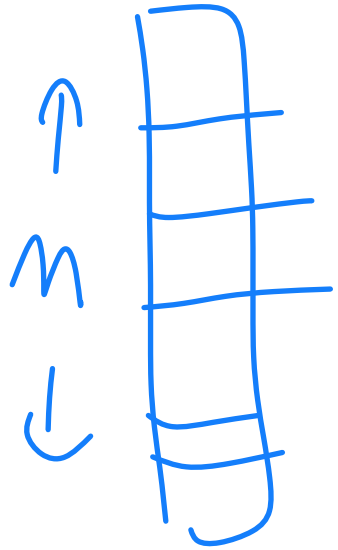
↳ conditional probability = prob

$$h(k_1 = x \mid h(k_2) = y) = h(k_1 = x)$$

Simple Uniform Hashing Assumption

Given table of size m, a simple uniform hash, h, implies

$$\forall k_1, k_2 \in U \text{ where } k_1 \neq k_2, \; Pr(h[k_1] = h[k_2]) = \frac{1}{m}$$



Uniform: All keys equally likely to hash to any position

$$Pr(h[k_1]) = \frac{1}{m}$$

$h(k_1)$ can have any value w/ equal prob

Independent: All key's hash values are independent of other keys

Separate Chaining Under SUHA

Table Size: m

Num objects: n

Claim: Under SUHA, expected length of chain is $\frac{n}{m}$

α_j = expected # of items hashing to position j

$$\alpha_j = \sum_i H_{i,j}$$

$$H_{i,j} = \begin{cases} 1 \text{ if item i hashes to j} \\ 0 \text{ otherwise} \end{cases}$$

$$E[\alpha_j] = \sum_i E[H_{i,j}]$$

prob of event × value of event

$$= \sum_i \Pr[H_{i,j}=1] \cdot 1 + \Pr[H_{i,j}=0] \cdot 0$$

$$E[\alpha_j] = \sum_{i}^{n \text{ items}} \Pr[H_{i,j}=1] = \text{by SUHA } \frac{n}{m}$$

$$\Pr[H_{i,j}=1] = \frac{1}{m}$$

Separate Chaining Under SUHA

Table Size: m

Num objects: n

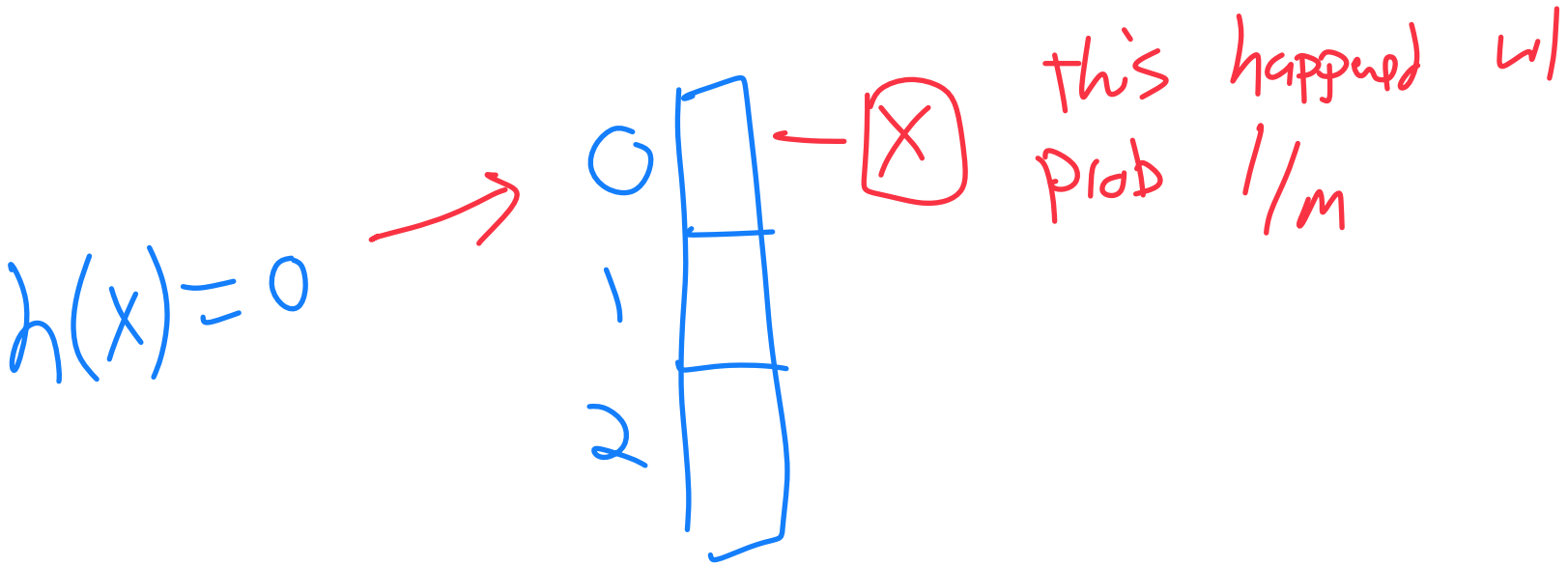
Claim: Under SUHA, expected length of chain is $\frac{n}{m}$

$\alpha_j =$ expected # of items hashing to position j

$$\alpha_j = \sum_i H_{i,j}$$

$$H_{i,j} = \begin{cases} 1 \text{ if item i hashes to j} \\ 0 \text{ otherwise} \end{cases}$$

$$E[\alpha_j] = E\left[\sum_i H_{i,j}\right]$$



Separate Chaining Under SUHA

Table Size: m

Num objects: n

Claim: Under SUHA, expected length of chain is $\frac{n}{m}$

$\alpha_j =$ expected # of items hashing to position j

$$\alpha_j = \sum_i H_{i,j}$$

$$H_{i,j} = \begin{cases} 1 \text{ if item i hashes to j} \\ 0 \text{ otherwise} \end{cases}$$

$$E[\alpha_j] = E[\sum_i H_{i,j}]$$

$$E[\alpha_j] = \sum_i Pr(H_{i,j} = 1) * 1 + Pr(H_{i,j} = 0) * 0$$

Separate Chaining Under SUHA

Table Size: m

Num objects: n

Claim: Under SUHA, expected length of chain is $\frac{n}{m}$

$\alpha_j =$ expected # of items hashing to position j

$$\alpha_j = \sum_i H_{i,j}$$

$$H_{i,j} = \begin{cases} 1 \text{ if item i hashes to j} \\ 0 \text{ otherwise} \end{cases}$$

$$E[\alpha_j] = E[\sum_i H_{i,j}]$$

$$E[\alpha_j] = \sum_i Pr(H_{i,j} = 1) * 1 + Pr(H_{i,j} = 0) * 0$$

$$E[\alpha_j] = n * Pr(H_{i,j} = 1)$$

Separate Chaining Under SUHA

Table Size: m

Num objects: n

Claim: Under SUHA, expected length of chain is $\frac{n}{m}$

$\alpha_j =$ expected # of items hashing to position j

$$\alpha_j = \sum_i H_{i,j}$$

$$H_{i,j} = \begin{cases} 1 \text{ if item i hashes to j} \\ 0 \text{ otherwise} \end{cases}$$

$$E[\alpha_j] = E[\sum_i H_{i,j}]$$

$$Pr[H_{i,j} = 1] = \frac{1}{m}$$

$$E[\alpha_j] = n * Pr(H_{i,j} = 1)$$

Separate Chaining Under SUHA

Claim: Under SUHA, expected length of chain is $\frac{n}{m}$

Table Size: m

Num objects: n

α_j = expected # of items hashing to position j

$$\alpha_j = \sum_i H_{i,j}$$

$$H_{i,j} = \begin{cases} 1 \text{ if item i hashes to j} \\ 0 \text{ otherwise} \end{cases}$$

$$E[\alpha_j] = E[\sum_i H_{i,j}]$$

sum of n independent events

$$E[\alpha_j] = n * Pr(H_{i,j} = 1)$$

$$Pr[H_{i,j} = 1] = \frac{1}{m}$$

uniformity

$$\mathbf{E[\alpha_j] = \frac{n}{m}}$$

load factor = $\frac{\text{num items}}{\text{number spaces}}$

Separate Chaining Under SUHA

Under SUHA, a hash table of size *m* and *n* elements:

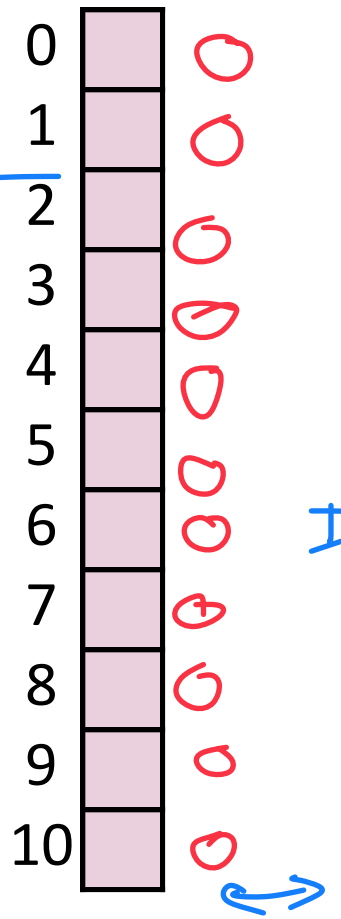
Find runs in: $1+\alpha$.

Insert runs in: 1.

Remove runs in: $1+\alpha$.

α is a constant that we control

$O(1)$**



as n grows this should move & more flat

In expectation size of each LL is n/m

$\alpha = n/m$

(Example of closed hashing)

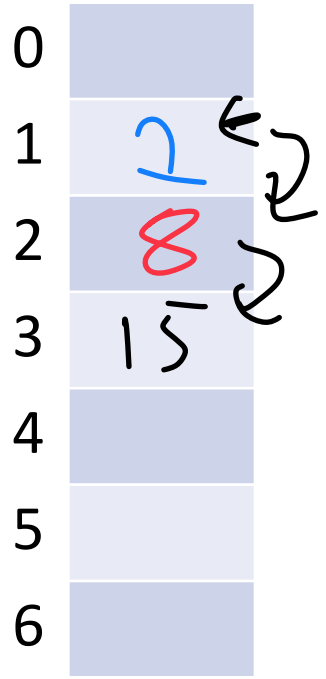
Collision Handling: Probe-based Hashing

S = { 1, 8 , 15}

h(k) = k % 7

|S| = n

|Array| = m



my array stores all data

Easiest next available add +1

1) hash input value

h(1) = 1

h(8) = 1

h(15) = 1

2) Place it at position in array

2.5) If collision, place at "next available" space

(Example of closed hashing)

Collision Handling: Linear Probing

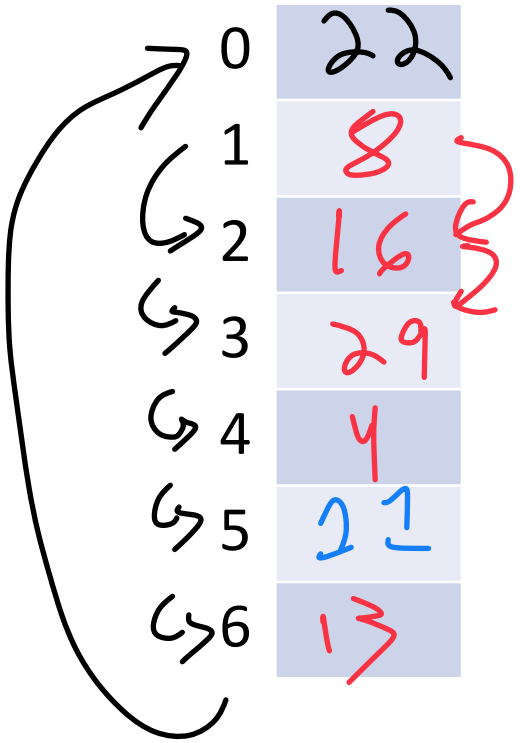
S = { 16, 8, 4, 13, 29, 11, 22 } **|S| = n**

h(k) = k % 7 **|Array| = m**

29%7 = 1

11%7 = 4

22%7 = 1



h(k, i) = (k + i) % 7

Try h(k) = (k + 0) % 7, if full…

Try h(k) = (k + 1) % 7, if full…

Try h(k) = (k + 2) % 7, if full…

Try …

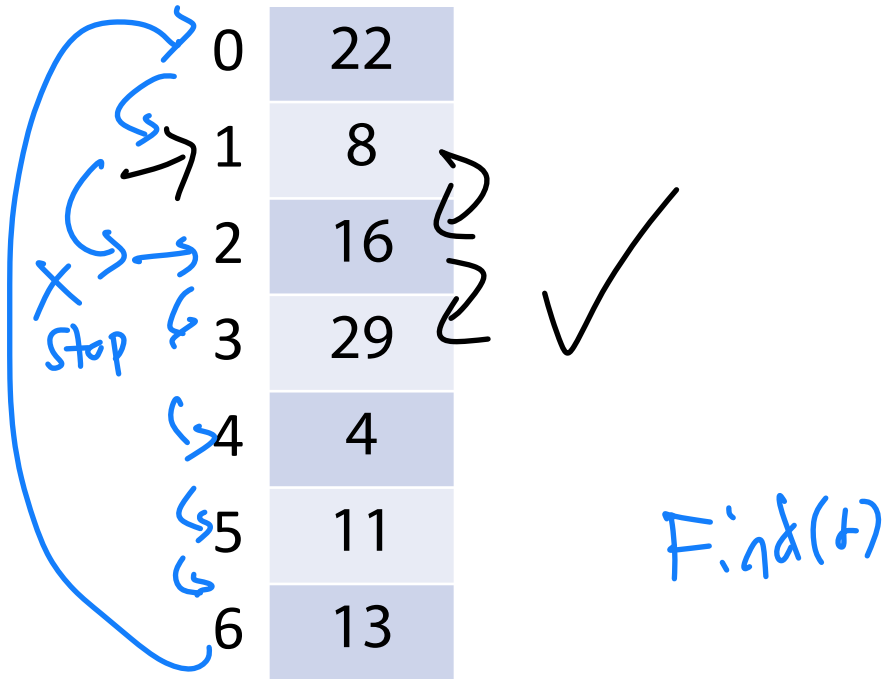
(Example of closed hashing)

Collision Handling: Linear Probing

$O(m) = O(n)$

S = { 16, 8, 4, 13, 29, 11, 22 } **|S| = n**

h(k, i) = (k + i) % 7 **|Array| = m**



Find(t)

`_find(29)`

1) H(29) = 1

2) If position not match, look at next available space

Stop if item is found
or we looped to start position
or we find a blank position (not tombstone)

(Example of closed hashing)

Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 } **|S| = n**

h(k, i) = (k + i) % 7 **|Array| = m**

Tradeoff:
To enable safe removal, need 1 bit per spot in array

0	22
1	8
2	~~16~~
3	29
4	4
5	11
6	13

Something was here

Search blank & stop

will find answer (w/ tombstone)

`_remove(16)`

1) Do find()

2) Remove item & tombstone position

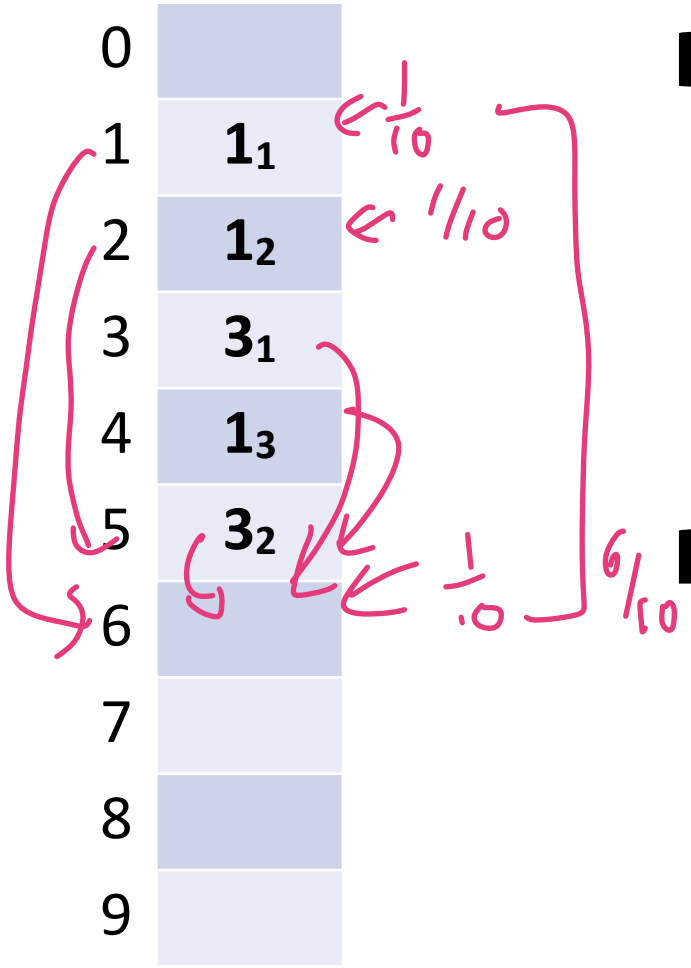
Find (29)

4 A 1 bit flag that something was here

A Problem w/ Linear Probing

Our Ht is SUHA
↳ uniform independent

Primary Clustering:



Description: Collisions create long runs of items

"Rich get richer"

We aren't being uniform!

Remedy: Pick a better "next available space"

(Example of closed hashing)

Collision Handling: Quadratic Probing

S = { 16, 8, 4, 13, 29, 12, 22 }

h(k) = k % 7

29 % 7 = 2

|S| = n

|Array| = m

0	12
1	8
2	16
3	22
4	4
5	29
6	13

12 % 7 = 5

+1

+4

h(k, i) = (k + i*i) % 7

Try h(k) = (k + 0) % 7, if full...

Try h(k) = (k + 1*1) % 7, if full...

Try h(k) = (k + 2*2) % 7, if full...

Try ...

5 + 7

5 + 4 % 7 = 2

5 + 9 = 14 % 7 = 0

9

16

22 % 7 = 1

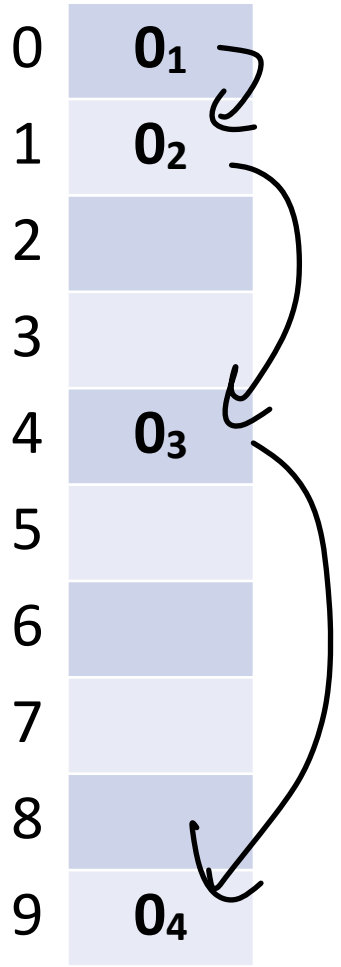
1 + 7

1 + 4

1 + 9 = 10 % 7 = 3

A Problem w/ Quadratic Probing

Secondary Clustering:



Description: Individual collisions still have long chains

Remedy: Be less consistent (but still deterministic)

Continue on Monday !!

(Example of closed hashing)

Collision Handling: Double Hashing

S = { 16, 8, 4, 13, 29, 11, 22 }

$h_1(k) = k \% 7$

$h_2(k) = 5 - (k \% 5)$

|S| = n

|Array| = m

0	
1	**8**
2	**16**
3	
4	**4**
5	
6	**13**

$h(k, i) = (h_1(k) + i*h_2(k)) \% 7$

Try $h(k) = (k + 0*h_2(k)) \% 7$, if full…

Try $h(k) = (k + 1*h_2(k)) \% 7$, if full…

Try $h(k) = (k + 2*h_2(k)) \% 7$, if full…

Try …

Running Times

(Expectation under SUHA)

(Don't memorize these equations, no need.)

Open Hashing:

insert: _____.

find/ remove: _____.

Closed Hashing:

insert: _____.

find/ remove: _____.

Running Times (Expectation under SUHA)

Open Hashing: $0 \leq \alpha \leq \infty$

insert: $\frac{1}{\quad}$.

find/ remove: $\frac{1+\alpha}{\quad}$.

Closed Hashing: $0 \leq \alpha < 1$

insert: $\frac{\frac{1}{\quad}}{1-\alpha}$.

find/ remove: $\frac{\frac{1}{\quad}}{1-\alpha}$.

Observe:

- As α increases:

Runtime approaches infinity

- If α is constant:

Runtime is constant

Running Times *(Don't memorize these equations, no need.)*

The expected number of probes for find(key) under SUHA

Linear Probing:

- Successful: **$\frac{1}{2}(1 + 1/(1-\alpha))$**
- Unsuccessful: **$\frac{1}{2}(1 + 1/(1-\alpha))^2$**

Double Hashing:

- Successful: **$1/\alpha * \ln(1/(1-\alpha))$**
- Unsuccessful: **$1/(1-\alpha)$**

Separate Chaining:

- Successful: **$1 + \alpha/2$**
- Unsuccessful: **$1 + \alpha$**

Instead, observe:

- As α increases:

Runtime approaches infinity

- If α is constant:

Runtime is constant

Running Times

The expected number of probes for find(key) under SUHA

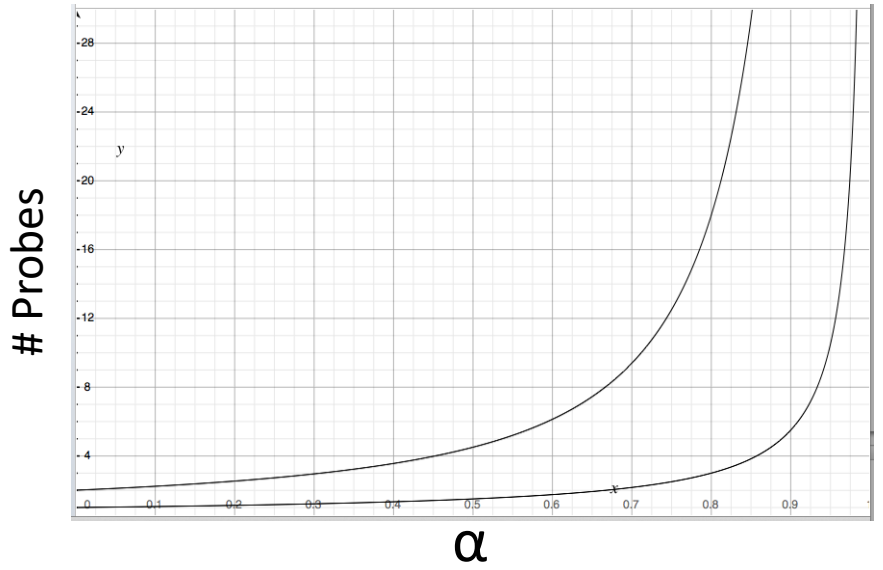
Linear Probing:

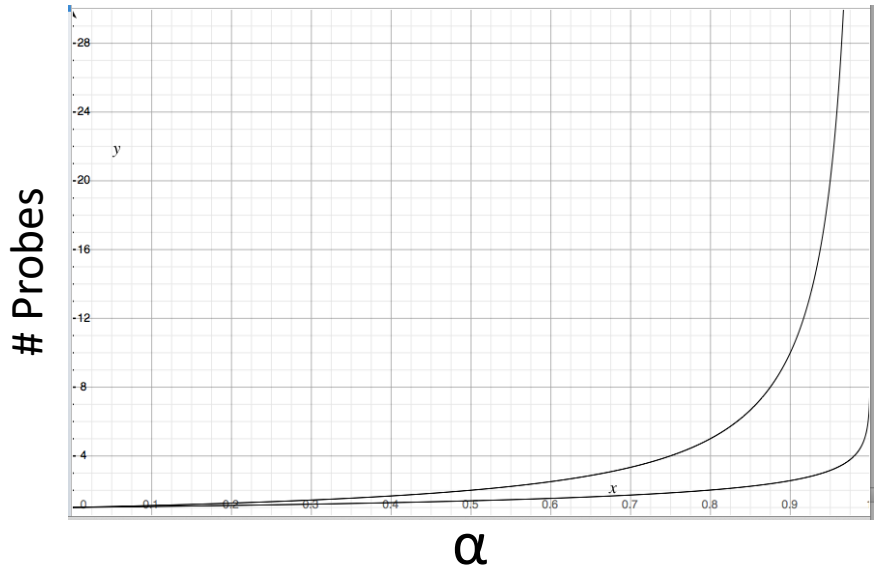
- Successful: **½(1 + 1/(1-α))**
- Unsuccessful: **½(1 + 1/(1-α))²**

Double Hashing:

- Successful: **1/α * ln(1/(1-α))**
- Unsuccessful: **1/(1-α)**

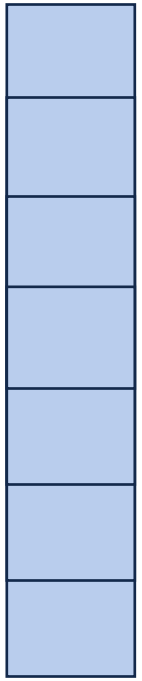
When do we resize?

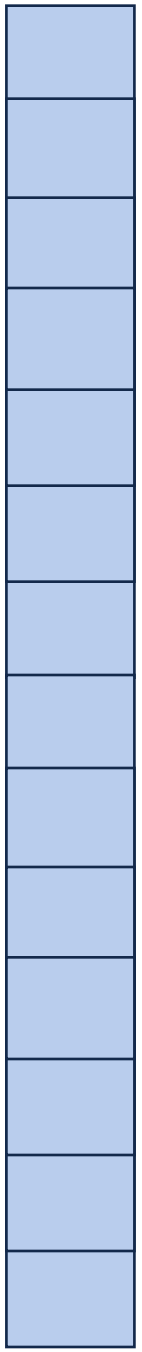




Resizing a hash table

How do you resize?





Which collision resolution strategy is better?

- Big Records:

- Structure Speed:

What structure do hash tables implement?

What constraint exists on hashing that doesn’t exist with BSTs?

Why talk about BSTs at all?

std::map in C++

```
T& map<K, V>::operator[]
pair<iterator, bool> map<K, V>::insert()
iterator map<K, V>::erase()


iterator map<K, V>::lower_bound( const K & );
iterator map<K, V>::upper_bound( const K & );
```

std::unordered_map in C++

```
T& unordered_map<K, V>::operator[]
pair<iterator, bool> unordered_map<K, V>::insert()
iterator unordered_map<K, V>::erase()
```

~~`iterator map<K, V>::lower_bound( const K & );`~~

~~`iterator map<K, V>::upper_bound( const K & );`~~

```
float unordered_map<K, V>::load_factor();
void unordered_map<K, V>::max_load_factor(float m);
```

Running Times

	Hash Table	AVL	Linked List
Find	Expectation*: Worst Case:		
Insert	Expectation*: Worst Case:		
Storage Space			