Data Structures and Algorithms
Probability in Computer Science

CS 225
Brad Solomon

November 10, 2025

Department of Computer Science

Learning Objectives

Formalize the concept of randomized algorithms

Review fundamentals of probability in computing

Distinguish the three main types of 'random' in computer science

Randomized Algorithms

A **randomized algorithm** is one which uses a source of randomness somewhere in its implementation.

detect presence of items

hash table

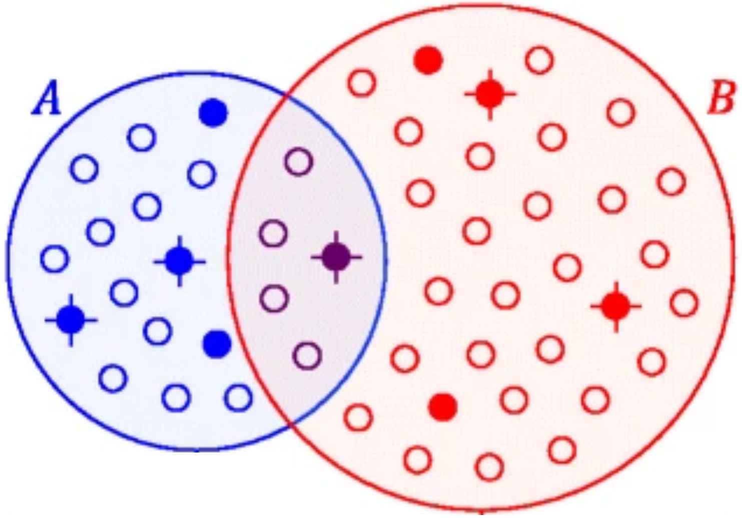
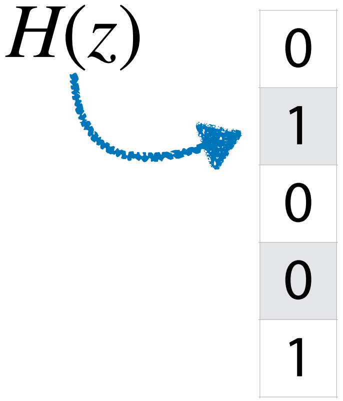
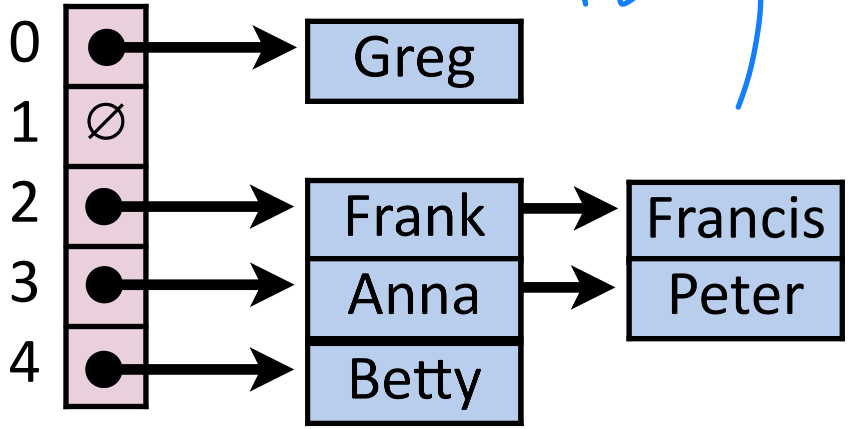


Figure from Ondov et al 2016

Find similarity





$H(x)$	0	2	1	0	0	4	0	2	0	6
$H(y)$	1	0	2	3	1	0	3	4	0	1
$H(z)$	2	1	0	2	0	1	0	0	7	2

compute counts

A faulty list

Imagine you have a list ADT implementation ***except***…

Every time you called **insert**, it would fail 50% of the time.

↳ Random summary (probabilistic data cleaning)

↳ we can filter out infrequent errors by chance

100x good things

1x bad thing

↳ Website caching

Quick Primes with Fermat's Primality Test

$[2, n-2]$

If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$

But… ***sometimes*** if n is composite and $a^{n-1} \equiv 1 \pmod{n}$

All prime #s $\equiv 1$

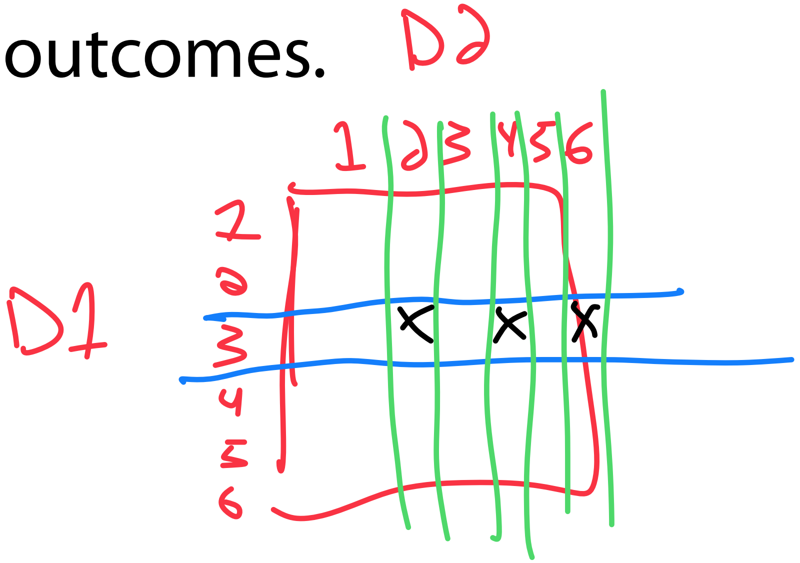
If $a=2$, $\frac{21853}{25 \cdot 10^9}$ which are composite but pass

Basically 0,00…%

Fundamentals of Probability

Imagine you roll a pair of six-sided dice.

The **sample space** Ω is the set of all possible outcomes.



An **event** $E \subseteq \Omega$ is any subset.

D1 = 3

&

D2 is even

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

A **random variable** is a function from events to numeric values.

↳ What is the expected dice roll value

"I roll two dice"

The **expectation** of a (discrete) random variable is:

$E[\text{two dice rolls}] = 2E[\text{one dice roll}]$

Prob of event · Value of event

two dice: $\frac{1}{36}(1+1) + \frac{2}{36}(1+2) + \ldots\ldots$

$$E[X] = \sum_{x \in \Omega} Pr\{X = x\} \cdot x$$

Prob of one dice: $\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + 3/6 + 4/6 + 5/6 + 6/6$

$= 3.5$

$E[\text{One dice roll}]$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$E[X + Y] = E[X] + E[Y]$ **(Claim)**

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$$E[X + Y] = E[X] + E[Y]$$

Prob of event

value of event

$$E[X + Y] = \sum_x \sum_y Pr\{X = x, Y = y\}(x + y)$$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

$$E[X+Y] = \sum_x \sum_y Pr\{X=x, Y=y\}(x+y)$$

$$= \sum_x x \cancel{\sum_y} Pr\{X=x, Y \cancel{=} y\} + \sum_y y \cancel{\sum_x} Pr\{X \cancel{=} x, Y=y\}$$

$$\sum_{\text{all events}} \text{prob} = 1$$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

$$E[X+Y] = \sum_x \sum_y Pr\{X = x, Y = y\}(x+y)$$

$$= \sum_x x \sum_y Pr\{X = x, Y = y\} + \sum_y y \sum_x Pr\{X = x, Y = y\}$$

E[X) E[Y]

$$= \sum_x x \cdot Pr\{X = x\} + \sum_y y \cdot Pr\{Y = y\}$$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$$E[X + Y] = E[X] + E[Y]$$

Randomization in Algorithms

1. Assume input data is random to estimate average-case performance

2. Use randomness inside algorithm to estimate expected running time

3. Use randomness inside algorithm to approximate solution in fixed time

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Claim: $S(n)$ is $O(n \ log \ n)$

N=3: AllBuild() with every possible permutation of insert order

1 2 3

1 3 2

2 1 3

2 3 1

3 2 1

3 1 2

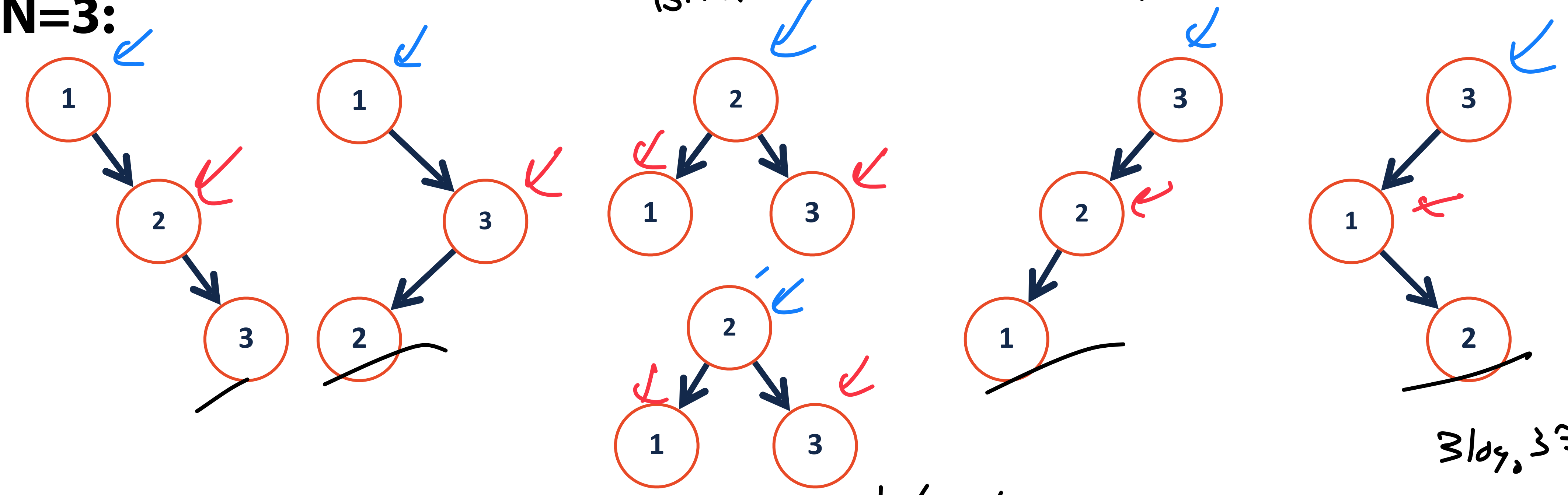
Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Claim: $S(n)$ is $O(n \ log \ n)$

Blue = path length 0
red path length 1
Black = length 2

N=3:



$3\log_2 3 \cong 4{,}75$

$6 \cdot 0 + 8 \cdot 1 + 4 \cdot 2 = 16/6 \sim 2{,}66$

Average-Case Analysis: BST

Let $S(n)$ be the **average** total internal path length **over all BSTs** that can be constructed by uniform random insertion of n objects

Let $0 \leq i \leq n - 1$ be the number of nodes in the left subtree.

Then for a fixed i, $S(n) = (n-1) + S(i) + S(n-i-1)$

$$S(n) = (n-1) + \frac{1}{n}\sum_{i=0}^{n-1} S(i) + S(n-i-1) \approx cn\ ln\ n$$

Largely skip!

Average-Case Analysis: BST

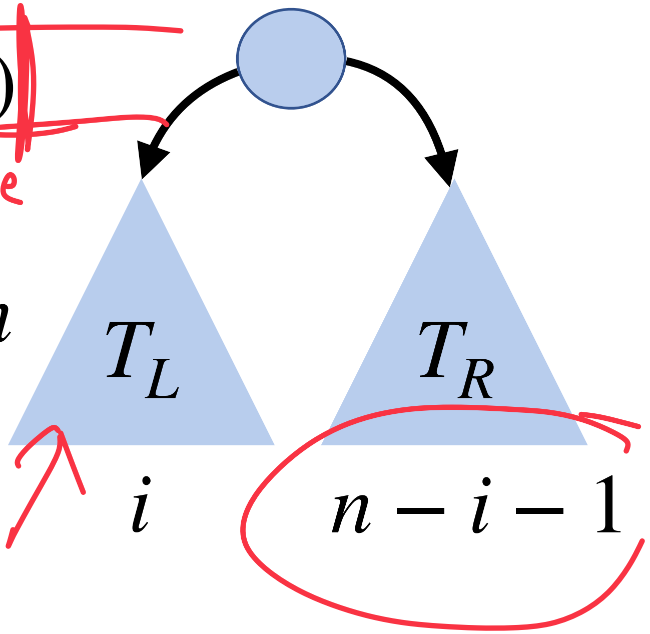
Let $S(n)$ be the **average** total internal path length **over all BSTs** that can be constructed by uniform random insertion of n objects

Let $0 \le i \le n-1$ be the number of nodes in the left subtree.

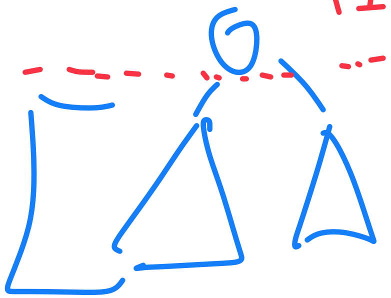
Then for a fixed i, $S(n) = (n-1) + \boxed{S(i) + S(n-i-1)}$

left & right subtree

$$S(n) = (n-1) + \frac{1}{n}\sum_{i=0}^{n-1} S(i) + S(n-i-1) \approx cn\ ln\ n$$



+1 to path length



n-1 nodes here

Here's a slide of math you should not bother learning (in the context of CS 225)

$$S(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} S(i)$$ (1) Guess recurrence form $S(i) = c * i\ ln(i)$

$$S(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} (ci\ ln\ i)$$ (2) Plug in recurrence

$$S(n) \leq (n-1) + \frac{2}{n}\int_{1}^{n} (cx\ ln\ x)dx$$ (3) $\sum_{i=1}^{n-1} f(i) \equiv \int_{1}^{n} f(x)dx$

$$S(n) \leq (n-1) + \frac{2}{n}\left(\frac{cn^2}{2}\ ln\ n - \frac{cn^2}{4} + \frac{c}{4}\right) \approx cn\ ln\ n$$

(4) $\int (cx\ lnx)\ dx$ can be expanded as shown above.

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

$S(n) \approx (n \;\; log \;\; n)$ is provable but a weak argument! **Why?**

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

$S(n) \approx (n\ log\ n)$ is provable but a weak argument! **Why?**

Our BST is average height log n

Randomness: Input dataset is considered random

Arguably to extend analysis to 'find' we also assume query is random.

Assumptions: Input dataset is uniform random in content and order

Same assumptions then extended to query

Randomization in Algorithms

1. Assume input data is random to estimate average-case performance

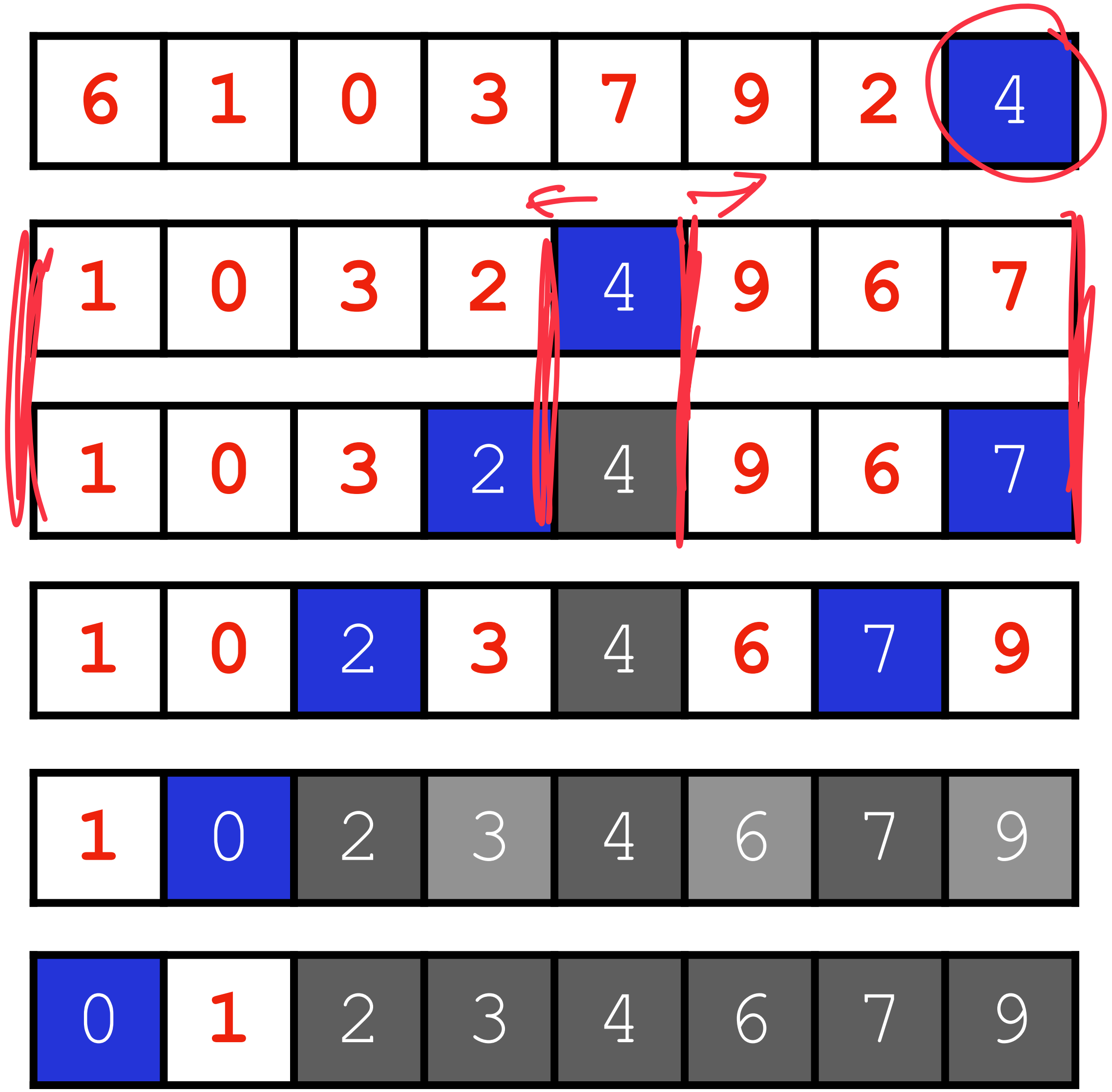
2. Use randomness inside algorithm to estimate expected running time

!! ↳ Alg might take a while but will work 100%

3. Use randomness inside algorithm to approximate solution in fixed time

↳ Alg runs fast but may not be correct

Quicksort Algorithm

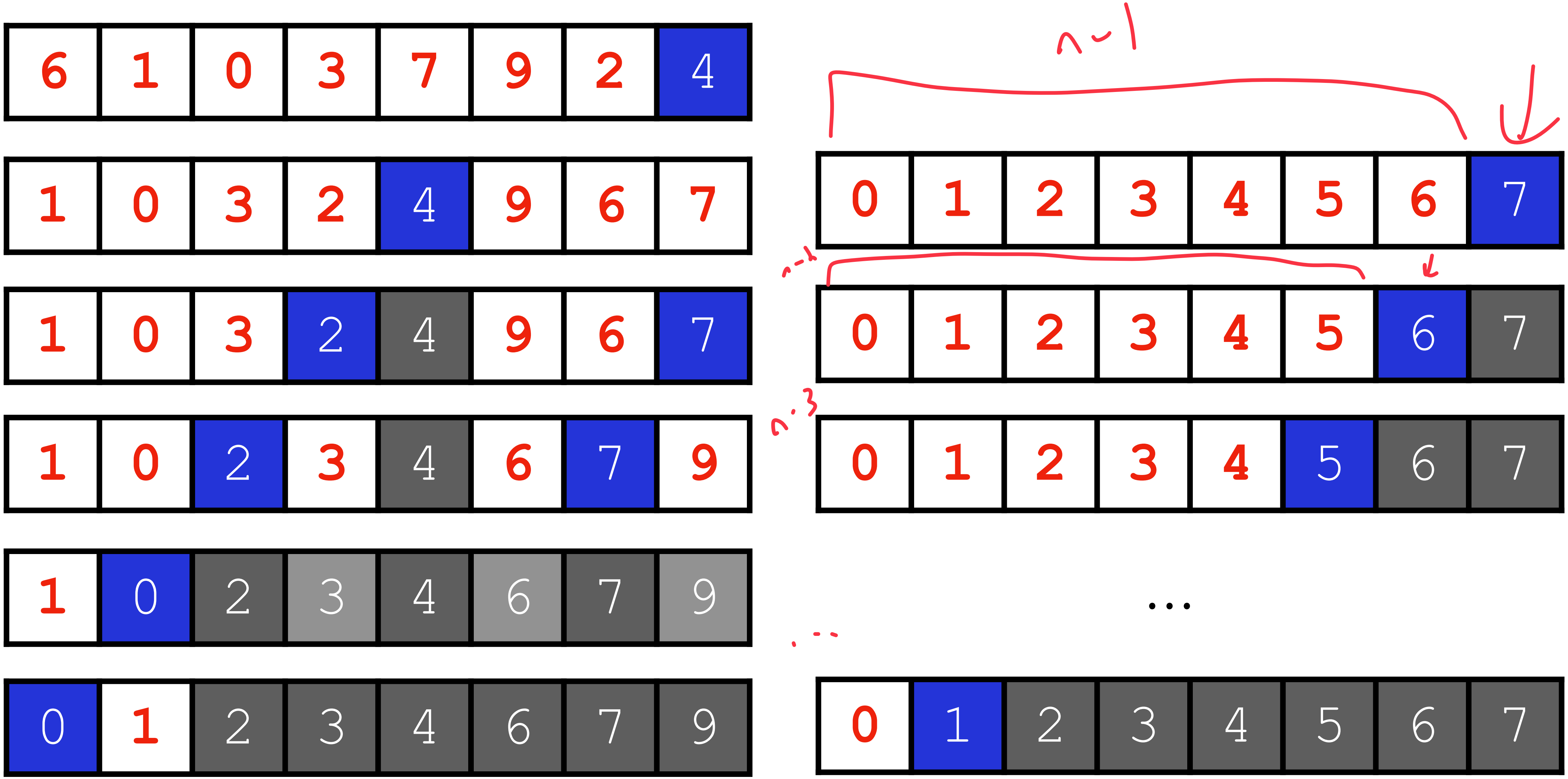


1) Pick Pivot (usually last item)

2) Split array around pivot

3) Recurse on partitions

Problem: Bad pivot leads to bad Big O!



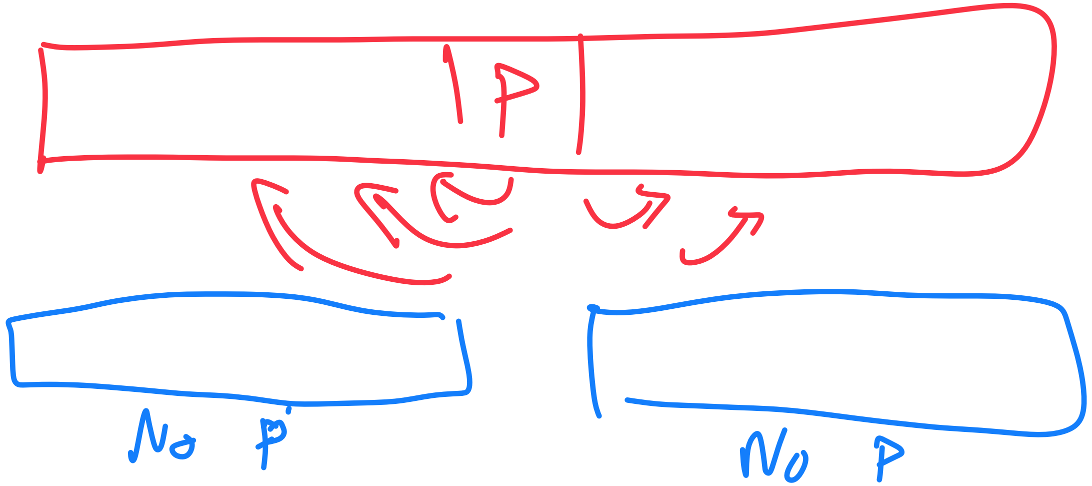
Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected time is $O(n\ log\ n)$ ***for any input!***

Key Idea: We never compare same pair twice!

Proof: Every comparison is against a pivot, but pivot not used in recursion



pivot compares to all items

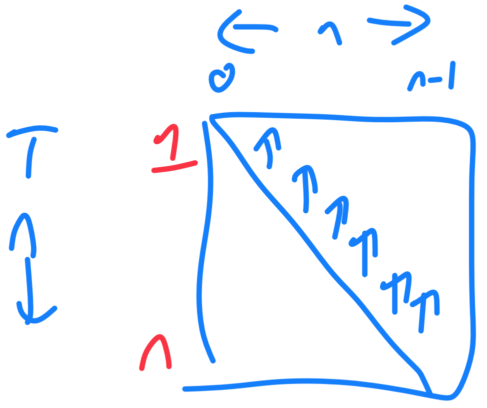
Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected time is $O(n \ log \ n)$ ***for any input!***

Let X be the total comparisons and X_{ij} be an **indicator variable**:

$$X_{ij} = \begin{cases} 1 & \text{if } i\text{th object compared to } j\text{th} \\ 0 & \text{if } i\text{th object not compared to } j\text{th} \end{cases}$$



1,2
2,7

Then…

$X =$ total # of comparisons $= \sum_i \sum_j X_{ij}$

$i < j$

$i+1 \ldots n$

This just makes math easier

Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected time is $O(n\ log\ n)$ ***for any input!***

Let X be the total comparisons and X_{ij} be an **indicator variable**:

$$X_{ij} = \begin{cases} 1 & \text{if } i\text{th object compared to } j\text{th} \\ 0 & \text{if } i\text{th object not compared to } j\text{th} \end{cases}$$

Then… $$X = \sum_{i}^{n} \sum_{j=i+1}^{n} X_{i,j}$$

We can prove that $E[X] = O(n\ log\ n)$ with a **proof by induction**!

Expectation Analysis: Randomized Quicksort

To show $E[X] = O(n\ log\ n)$, we need to first get $E[X_{i,j}]$

Claim: $E[X_{i,j}] = \dfrac{2}{j - i + 1}$.

$$\frac{2}{i+1-i+1} = \frac{2}{2} = 1$$

Base Case: (N=2)

A	B
i	j

IF A is pivot:

$A \to B$ 1 comparison

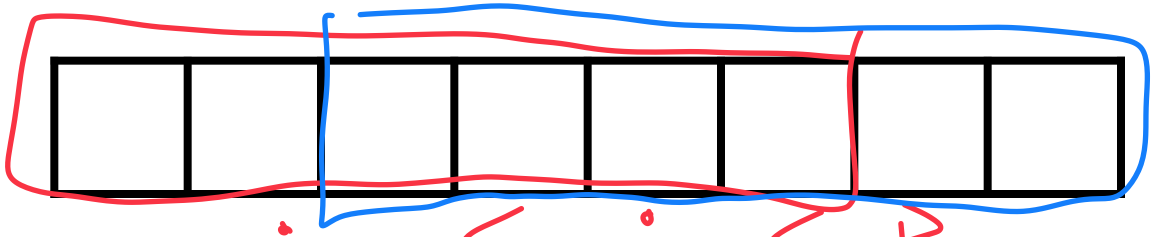
B is pivot

$B \to A$ 1 comparison

Expectation Analysis: Randomized Quicksort

$i < j$

Claim: $E[X_{i,j}] = \dfrac{2}{j - i + 1}$ **Induction:** Assume true for all inputs of $< n$



$i < j < p$

$p < i < j$

$i \le p \le j$

$= \Pr[X_{ij} = 1 \mid j < p] \cdot \Pr[j < p]$

$+ \Pr[X_{ij} = 1 \mid p < i] \cdot \Pr[p < i]$

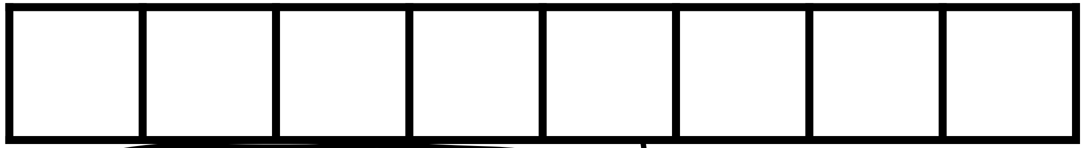
$+ \Pr[X_{ij} = 1 \mid i \le p \le j] \cdot \Pr[i \le p \le j]$

* Only possible if $i = p$ or $j = p$

Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \dfrac{2}{j - i + 1}$

Induction: Assume true for all inputs of $< n$



$Pr[X_{ij} \mid j < p] * Pr[j < p] +$

↳ By IH $\frac{2}{j-i+1}$

i < j < p

↳ smaller than n partition

$Pr[X_{ij} \mid i > p] * Pr[i > p] +$

↳ By IH $\frac{2}{j-i+1}$

p < i < j

$Pr[X_{ij} \mid i < p < j] * Pr[i < p < j]$

i ≤ p ≤ j

$\frac{2}{j-i+1}$

↳ we only compare i & j if i=p or j=p

↳ All other cases, they are in different partitions

Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \dfrac{2}{j-i+1}$ **Induction:** Assume true for all inputs of $< n$

$Pr[X_{ij} \mid j < p] * Pr[j < p] +$

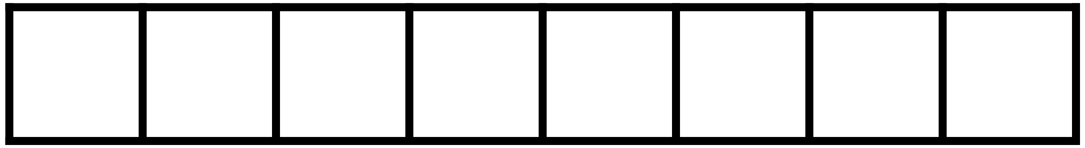
By IH, $\dfrac{2}{j-i+1}$

$Pr[X_{ij} \mid i > p] * Pr[i > p] +$

By IH, $\dfrac{2}{j-i+1}$

$Pr[X_{ij} \mid i < p < j] * Pr[i < p < j]$

Pivot must be either i or j — happens twice so $\dfrac{2}{j-i+1}$



i < j < p

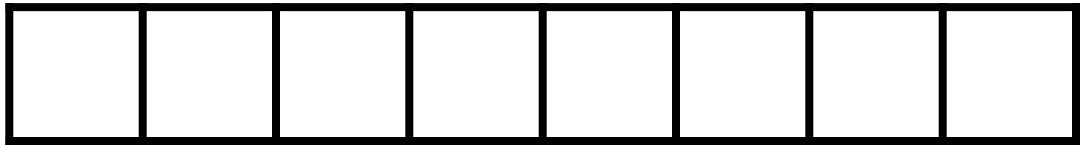
p < i < j

i ≤ p ≤ j

Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \dfrac{2}{j-i+1}$

Induction: Assume true for all inputs of $< n$



$= 1$

We can rewrite as: $\dfrac{2}{j-i+1} * \left(Pr[j < p] + Pr[i > p] + Pr[i \le p \le j]\right)$

Pull out $\dfrac{2}{j-i+1}$

This is every possible event in universe

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j - i + 1}$$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n} 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-i+1}\right)$$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n} 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-i+1}\right)$$

$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1) \leq 2n \cdot H_n \leq 2n\ ln\ n$$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j - i + 1}$$

$$E[X] = \sum_{i=1}^{n} 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n - i + 1}\right)$$

(1) Expand out inner sum

$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1)$$

(2) $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$

$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1) \leq 2n \cdot H_n \leq 2n \ ln \ n$$

(3) $H_n = \theta(log \ n)$

Expectation Analysis: Randomized Quicksort

Summary: Randomized quick sort is $O(n\ \ log\ \ n)$ regardless of input

Randomness:

Assumptions:

Expectation Analysis: Randomized Quicksort

Summary: Randomized quick sort is $O(n\ log\ n)$ regardless of input

Randomness: The choice of pivot at each step

The analysis here works for any choice of input dataset!

Assumptions: Only that random numbers are actually random

While strictly not true, generally an acceptable assumption in practice

Ex: Park, Kyung Hwan, et al. "High rate true random number generator using beta radiation." AIP Conference Proceedings. Vol. 2295. No. 1. AIP Publishing LLC, 2020.

Randomization in Algorithms

1. Assume input data is random to estimate average-case performance

2. Use randomness inside algorithm to estimate expected running time

3. Use randomness inside algorithm to approximate solution in fixed time

Probabilistic Accuracy: Fermat primality test

Pick a random a in the range $[2, \ p-2]$

If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$

But… ***sometimes*** if n is composite and $a^{n-1} \equiv 1 \pmod{n}$

Probabilistic Accuracy: Fermat primality test

The outcome label

O(1)

	$a^{p-1} \equiv 1 \pmod{p}$ says prime	$a^{p-1} \not\equiv 1 \pmod{p}$ says not prime
p is prime	100% :)	0% :)
p is not prime	Some FPR δ = FPR error!	$1-\delta$:)

Probabilistic Accuracy: Fermat primality test

FPR

Let's assume $\alpha = .5$

First trial: $a = a_0$ and prime test returns 'prime!'

Second trial: $a = a_1$ and prime test returns 'prime!'

Third trial: $a = a_2$ and prime test returns 'not prime!'

Repeated random trials

Is our number prime?

↳ Not prime!

B/c 100% if not prime is seen once it's not prime

What is our **false positive** probability? Our **false negative** probability?

$(.5)^3 = 0.125$

Probabilistic Accuracy: Fermat primality test

Summary: Randomized algorithms can also have fixed (or bounded) runtimes at the cost of probabilistic accuracy.

Randomness:

Didnt get to but its simple
(see next slide :))

Assumptions:

Probabilistic Accuracy: Fermat primality test

Summary: Randomized algorithms can also have fixed (or bounded) runtimes at the cost of probabilistic accuracy.

Randomness: The choice of α.

We can even pick more than one α if we want!

Assumptions: Only that random numbers are actually random

While strictly not true, generally an acceptable assumption in practice

Types of randomized algorithms

A **Las Vegas** algorithm is a randomized algorithm which will always give correct answer if run enough times but has no fixed runtime.

A **Monte Carlo** algorithm is a randomized algorithm which will run a fixed number of iterations and may give the correct answer.

What type of algorithm is Fermat's primality test?

What type of algorithm is randomized quick sort?

Next Class: Randomized Data Structures

Sometimes a data structure can be **too ordered / too structured**

Randomized data structures rely on **expected** performance

Randomized data structures 'cheat' tradeoffs!