Data Structures

Single Source Shortest Path

CS 225
Brad Solomon

November 5, 2025

UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN
Department of Computer Science


YOU ARE INVITED TO
TRICK OR RESEARCH
RESEARCH FAIR, POSTER SESSION, PARTICIPATING LABS, AND CANDY!
WEDNESDAY, NOVEMBER 5, 2025
4:00 P.M. TO 5:30 P.M.
Siebel School of Computing and Data Science
Be there and be scared!
CS STARS
STUDENT AMBASSADORS/RESEARCH SCHOLARS

Exam 4 (11/12 — 11/14)

Autograded MC and one coding question

Manually graded short answer prompt

Practice exam will be on PL

Topics covered can be found on website

Registration is open!

https://courses.engr.illinois.edu/cs225/fa2025/exams/

Learning Objectives

Finish discussion about Prim's runtime

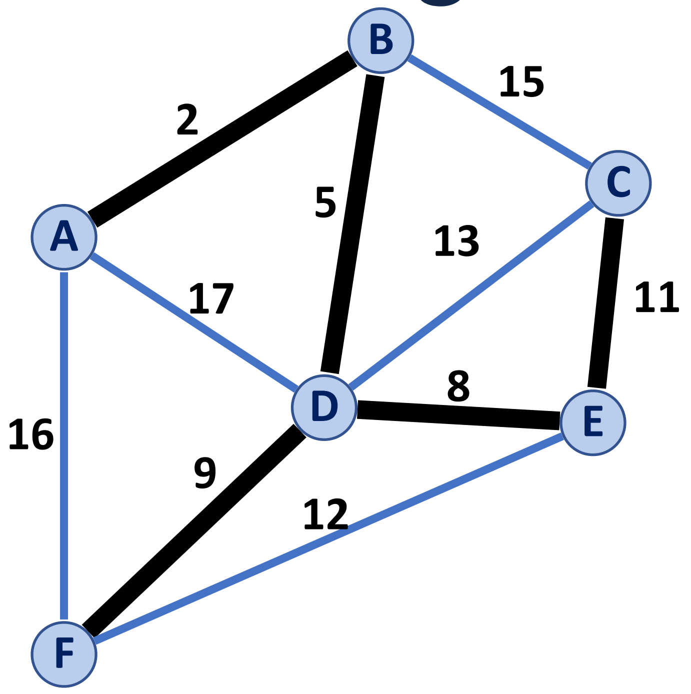
Compare Kruskal and Prim MST Algorithms

Introduce Single-Source Shortest Path Problem

Discuss Dijkstra's Algorithm

Extend to All-Paths Shortest Path (if time)

Prim's Algorithm



0 ∞ ∞ ∞ ∞ ∞

A	B	C	D	E	F
0, —	2, A	11, E	5, B	8, D	9, D

```
PrimMST(G, s):
  Input: G, Graph;
         s, vertex in G, starting vertex
  Output: T, a minimum spanning tree (MST) of G

  foreach (Vertex v : G.vertices()):
    d[v] = +inf
    p[v] = NULL
  d[s] = 0

  PriorityQueue Q   // min distance, defined by d[v]
  Q.buildHeap(G.vertices())
  Graph T           // "labeled set"

  repeat n times:
    Vertex m = Q.removeMin()
    T.add(m)
    foreach (Vertex v : neighbors of m not in T):
      if cost(v, m) < d[v]:
        d[v] = cost(v, m)
        p[v] = m

  return T
```

Init

pick next min edge

update the distances

Prim's Big O

$|V| = n, |E| = m$

7 — 9: O(n)

12—14:

MinHeap: O(n)

Unsorted Array: O(1)

16—22: Complicated!

```
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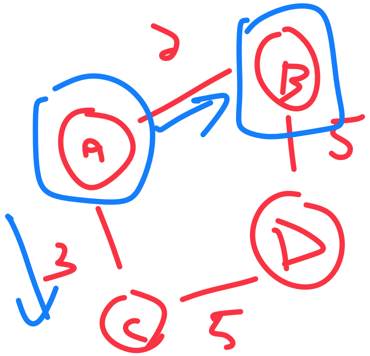
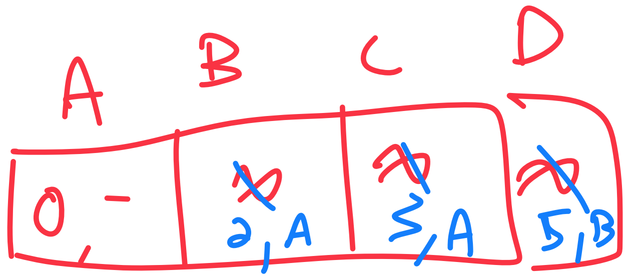
23
```

Depends on choice of **PriorityQueue** (MinHeap vs Unsorted Array)

Depends on choice of **Graph** (Adjacency Matrix vs Adjacency List)

Prim's Big O



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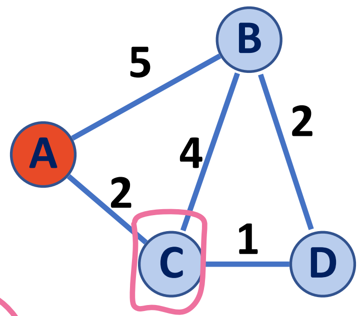
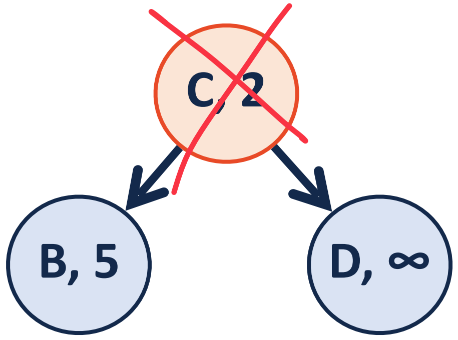
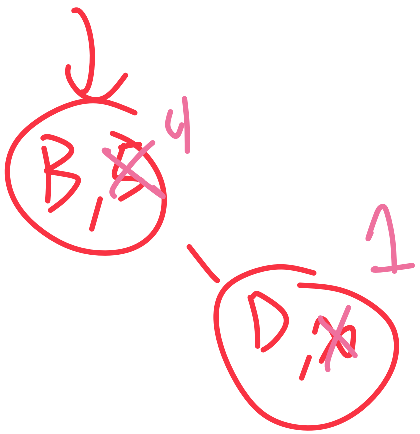
```

A	B	C	D
0	~~5~~ 4	2	~~∞~~ 1





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```

$O(n)$

$O(\log n)$

update value is $O(1)$
heapify to restore heap
$O(\log n)$

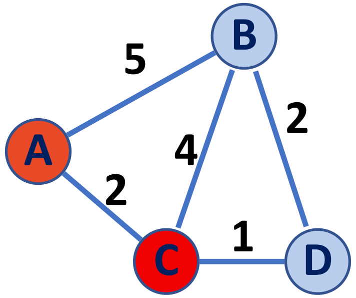
up to m times

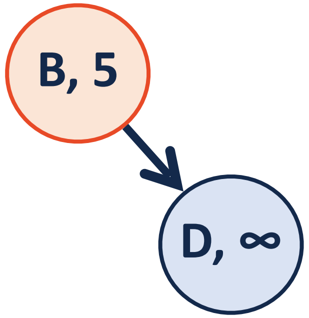
same logic

$\sum \deg(v) = 2m$

	Adj. Matrix	Adj. List
Heap	O(n) + $O(n \log n)$ + O(n^2) + $O(m \log n)$	O(n) + $O(n \log n)$ + O(m) + $O(m \log n)$

A	B	C	D
0	5	2, A	∞



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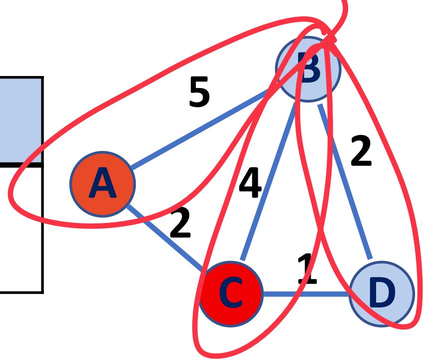
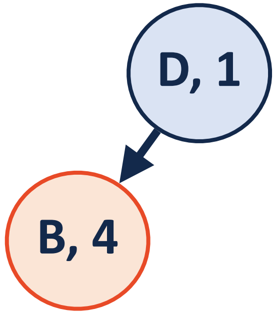
```

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + ________________	O(n) + O(n log n) + O(m) + ________________

A	B	C	D
0	4	2, A	1



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```

O(n)

1) Change minheap value

2) HeapifyUp()

Build a heap

remove min

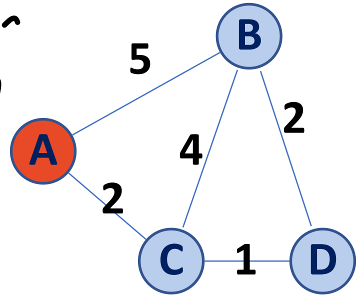
Variable cost to lookup neighbors

# of times we need to change

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + O(m log n)	O(n) + O(n log n) + O(m) + O(m log n)

(A, 0)
(D, ∞)
(C, 2)
(B, 5)

remove min by walking full length of array



Length of array? O(n)

next timestep

D, ∞
B, 5

$n + n-1 + (n-2) + \dots 1$

```
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        d[v] = cost(v, m)
        p[v] = m
```

$O(1)$

$O(n)$

$O(1)$

$|V| = n$ $|E| = m$

	Adj. Matrix	Adj. List
Heap	O(n² + m lg(n))	O(n lg(n) + m lg(n))
Unsorted Array	$O(n^2)$	$O(n^2)$

Prim's Algorithm

Sparse Graph: m ~ n

Adj List Heap best

Dense Graph: m ~ n^2

Unsorted Array best

```
PrimMST(G, s):
  foreach (Vertex v : G.vertices()):
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```

O(m)

sparse graph

	Adj. Matrix	Adj. List
Heap	$O(n^2 + m\lg(n))$	$\boxed{O(n\lg(n))} + \cancel{m}^{\,n}\lg(n))$ ← $n^2 \log n$
Unsorted Array	$O(n^2)$ + O(n^2) dense	best! $O(n^2)$ + O(m)

:(

MST Algorithm Runtime:

Kruskal's Algorithm:

O(n + m log (n))

m

Prim's Algorithm:

O(n log(n) + m log (n))

Sparse Graph: m ~ n

$n + n \log n$

$n \log n + n \log n$

Same on both!

Dense Graph: m ~ n²

$n + n^2 \log n$

$n \log n + n^2 \log n$

Suppose I have a new heap:

	Binary Heap	Fibonacci Heap
Remove Min	O(lg(n))	O(lg(n))
Decrease Key	O(lg(n))	O(1)* !!

What's Prim's updated running time?

$O(n \log n + m)$

```
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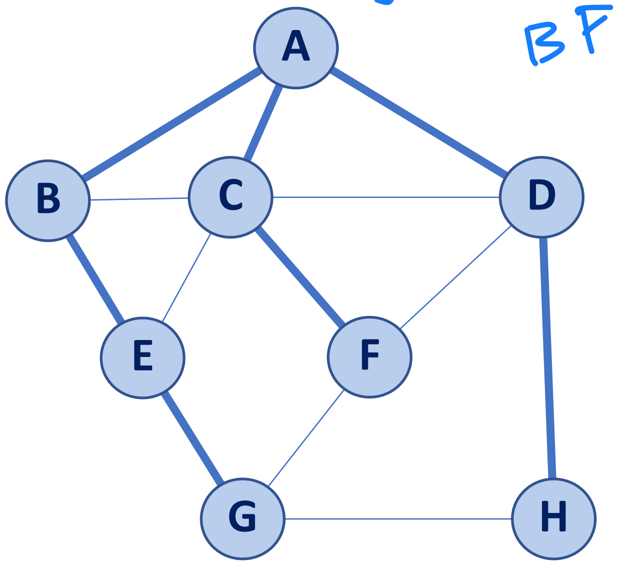
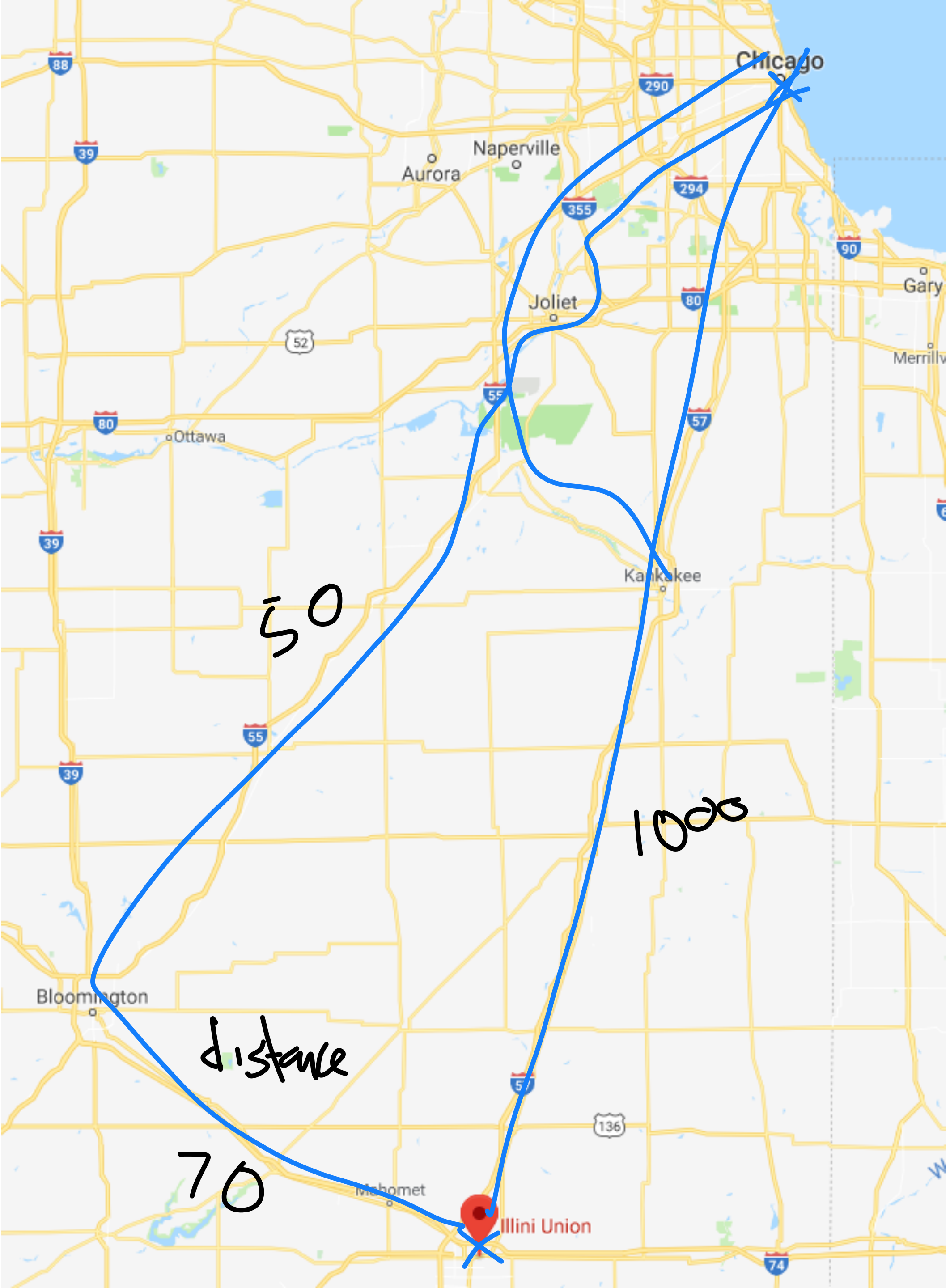
        p[v] = m
```

← O(1)*] A total of m times

Shortest Path

If unweighted use BFS!

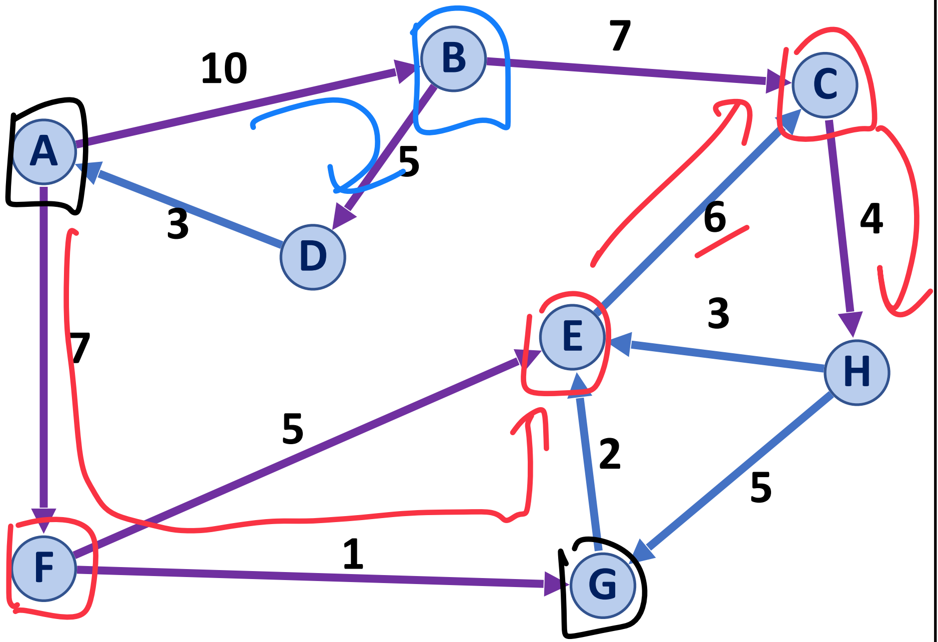




Dijkstra's Algorithm (SSSP)

Very similar to Prim!

from start (A) to all other nodes

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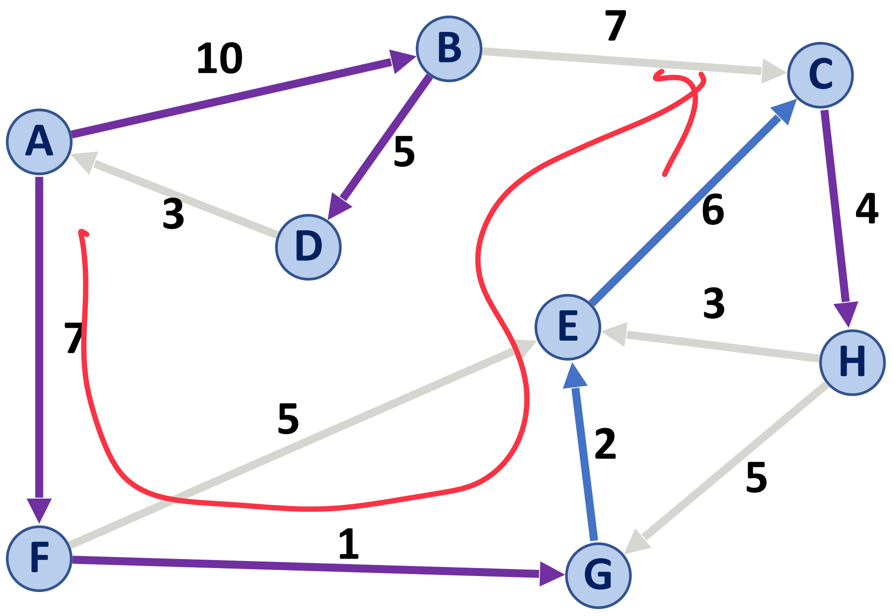
      if cost(u,v) + d[u] < d[v]:
        d[v] = cost(u,v) + d[u]
        p[v] = u
```

Initialize

Is path through u plus edge u,v better than current best path to v?

~~A~~	B	C	D	E	~~F~~	~~G~~	H
--	A	~~B~~ E	B	~~F~~ G	A	F	C
0	~~∞~~ 10	~~∞~~ ~~17~~ 16	~~∞~~ 15	~~∞~~ ~~12~~ 10	~~∞~~ 7	~~∞~~ 8	~~∞~~ 20

Dijkstra's Algorithm (SSSP)



This is not MST!

```
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```

A	B	C	D	E	F	G	H
--	A	E	B	G	A	F	C
0	10	16	15	10	7	8	20

This solves shortest path

Dijkstra's Algorithm (SSSP)

Prim's running time?

What is the running time of Dijkstra's Algorithm?

$n \log n + m \log m$ using minheap

using fib heap

$O(n \log n + m)$

multiple valid runtimes (weighted)

vs

BFS unweighted

$O(n + m)$

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        p[v] = m

  return T
```

this is only change

Dijkstra’s Algorithm (SSSP)

What is the running time of Dijkstra’s Algorithm?

O(m + n log n)

The same as Prim's!

6-9: O(n)

11-12: O(n)

15: repeat below n x

16-22: O(log n)

[w/ Fib Heap O(1) updates]

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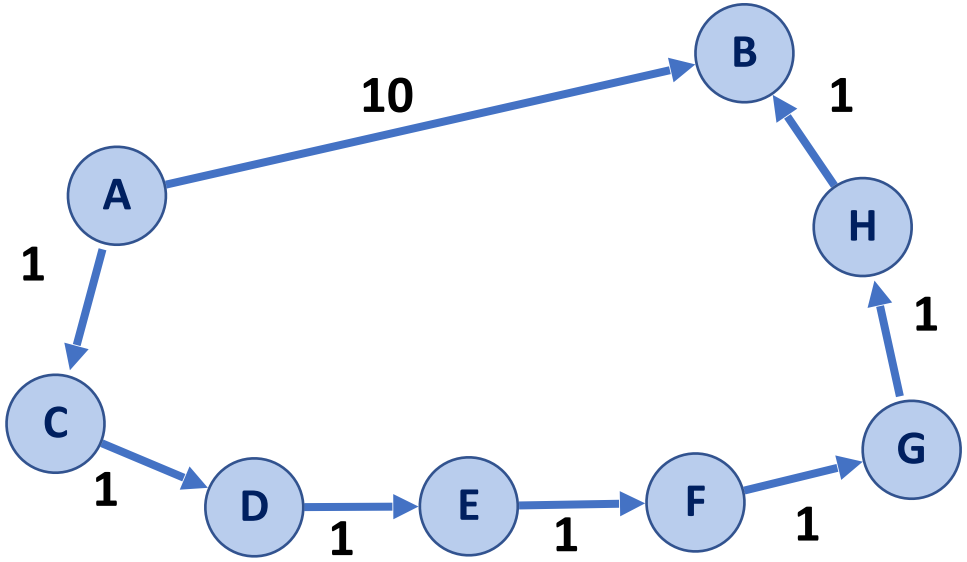
23   return T
```

Dijkstra’s Algorithm (SSSP)

Claim: Dijkstras will always visit a node through its optimal shortest path.

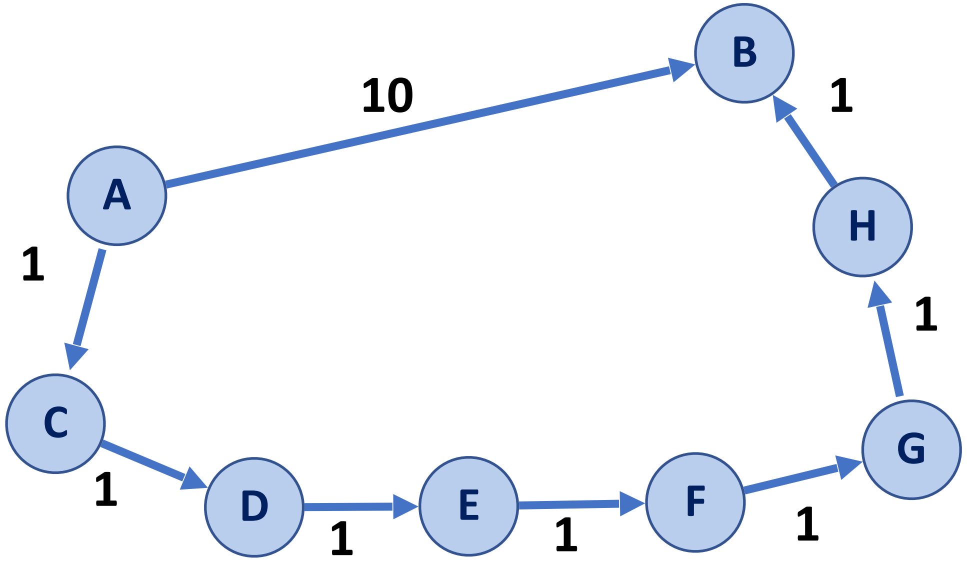
When we will visit B in the following graph?



Dijkstra's Algorithm (SSSP)

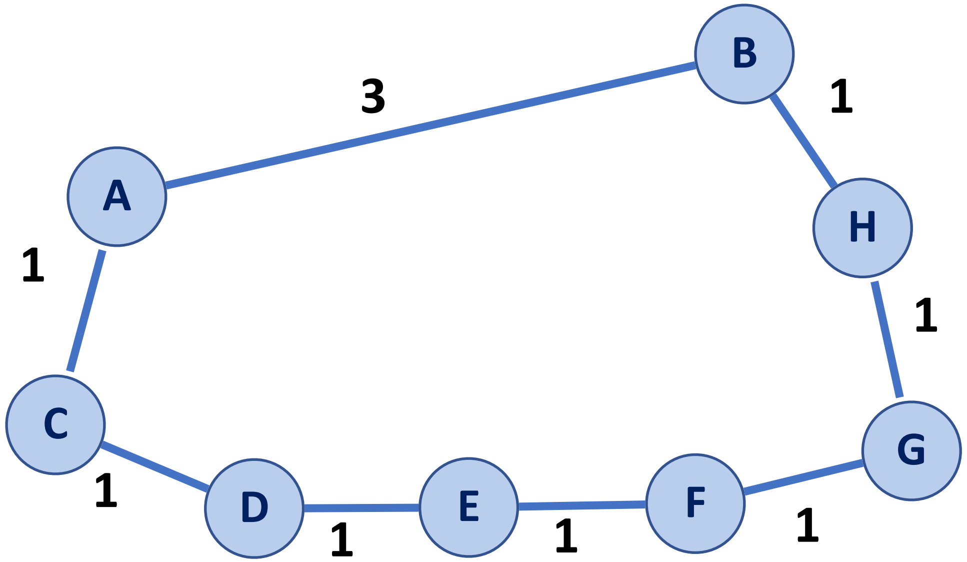
Claim: Dijkstras will always visit a node through its optimal shortest path.

When we will visit H in the following graph?



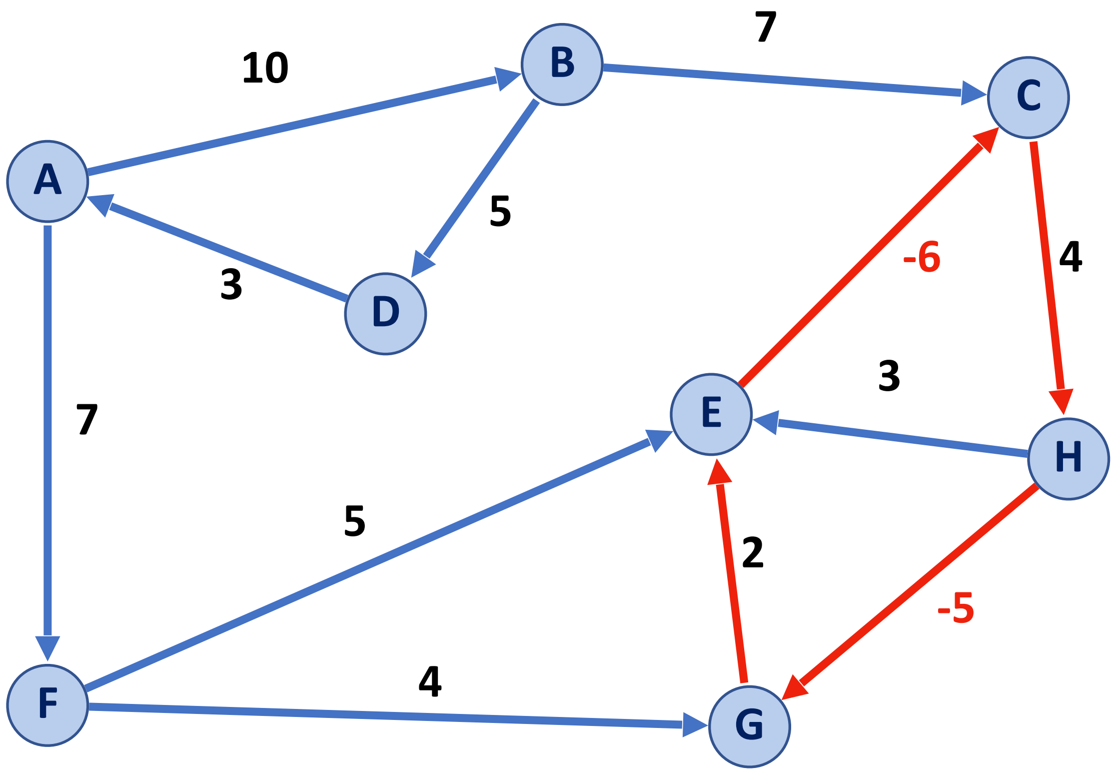
Dijkstra's Algorithm (SSSP)

How does Dijkstra's algorithm handle undirected graphs?



Dijkstra’s Algorithm (SSSP)

How does Dijkstras handle a negative weight cycle?



Dijkstra's Algorithm (SSSP)

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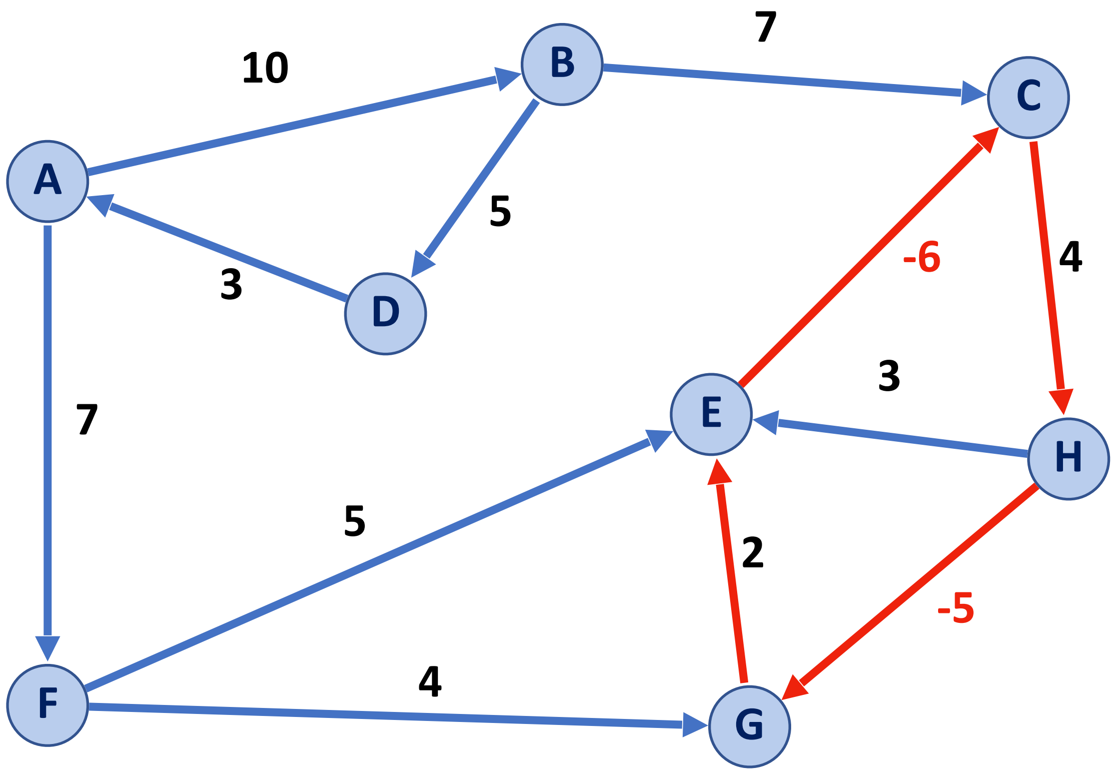


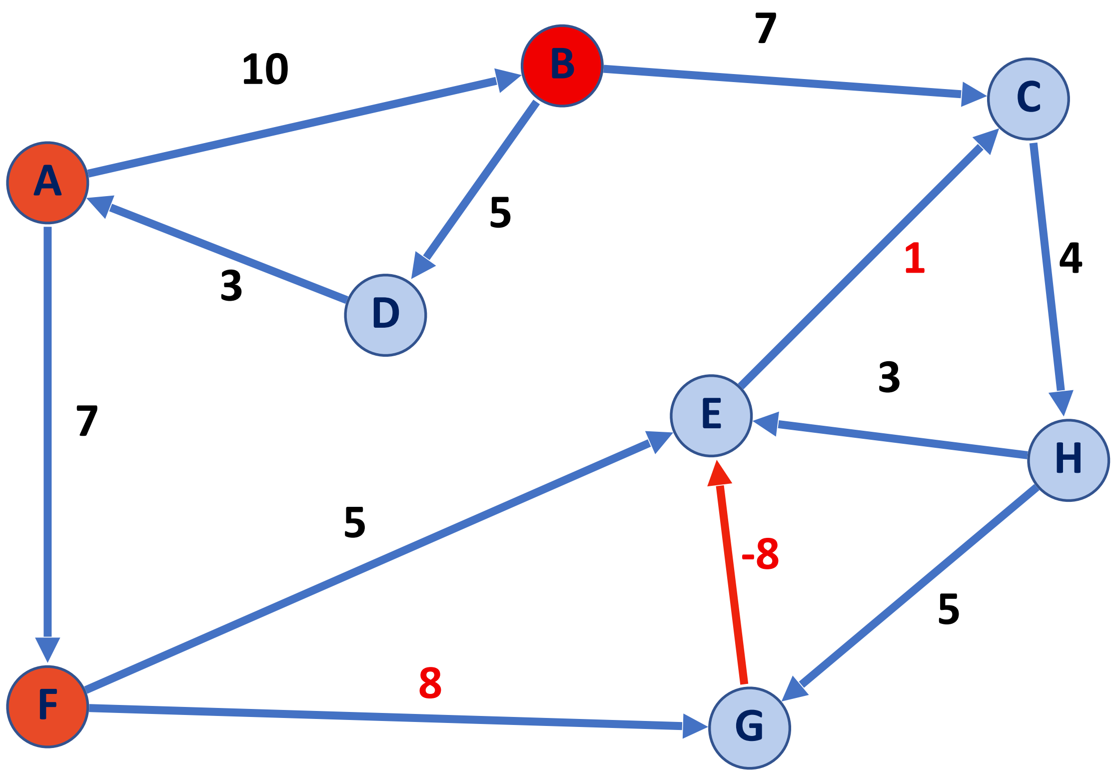
Shortest Path (A → E): A → F → E → (C → H → G → E)*

Length: 12

Length: -5 (repeatable)

Dijkstra’s Algorithm (SSSP)

How does Dijkstras handle a negative weight edge without a cycle?



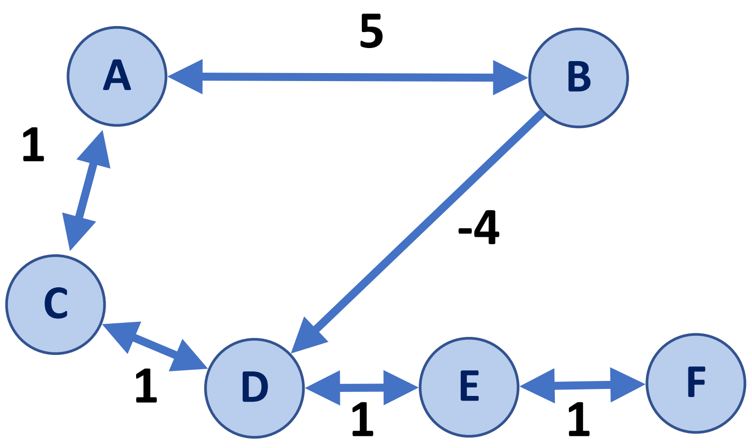
A	B	C	D	E	F	G	H
--	A	B	B	F	A	F	--
0	10	17	15	12	7	15	∞

Dijkstra's Algorithm (SSSP)

We assume that item pulled out of priority queue is **the next smallest item**

Negative weights break this assumption!

A	B	C	D	E	F
--					
0					



Dijkstra's Algorithm (SSSP)

Recalculating all distances is possible, but algorithm runtime is very bad!

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  repeat until Q.empty():
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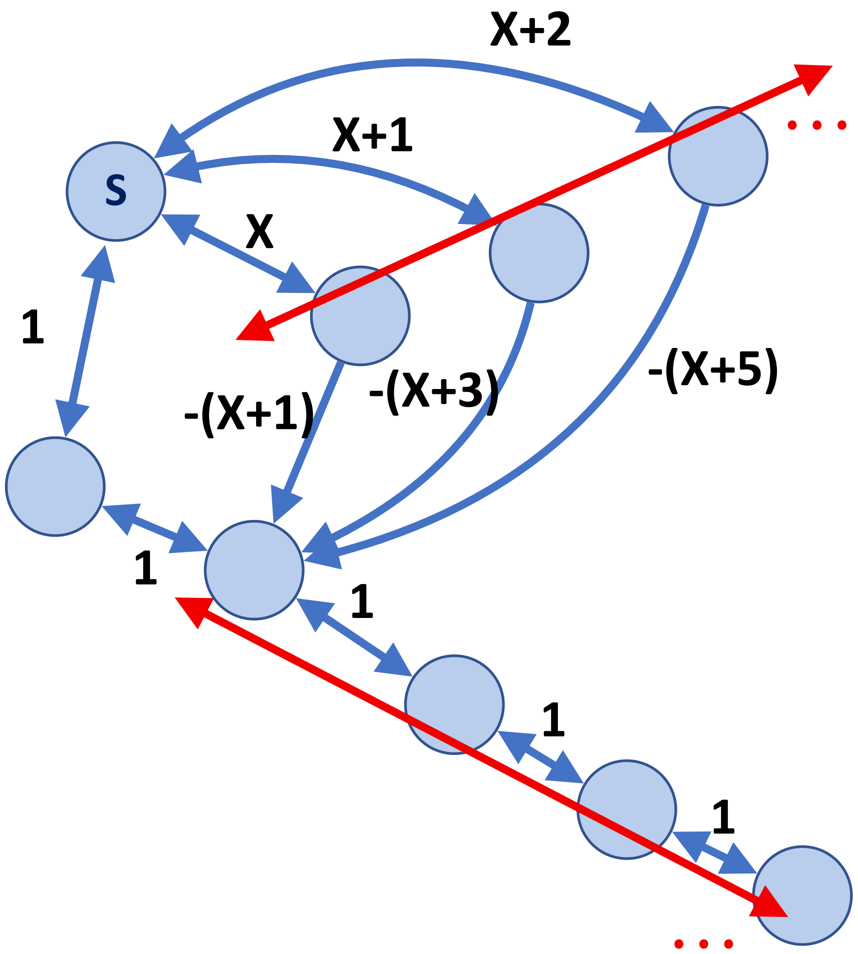
    T.add(u)
    foreach (Vertex v : neighbors of u not in T):
      if cost(u, v) + d[u] < d[v]:
        d[v] = cost(u, v) + d[u]
        p[v] = m
        if v not in Q:
           Q.push(v)
  return T
```

Dijkstra’s Algorithm (SSSP)

Recalculating all distances is possible, but algorithm runtime is very bad!

Worst case: ~n/2 nodes each updating ~n/2 nodes distances

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          Q.push(v)
  return T
```

Dijkstra's Algorithm (SSSP)

Dijkstras Algorithm works only on non-negative weights

Optimal implementation:

Fibonacci Heap

If dense, unsorted list ties

Optimal runtime:

Sparse: $O(m + n \log n)$

Dense: $O(n^2)$

```
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```

Landmark Path Problem

What if I wanted to get the shortest path from A to G but stopping at L along the way?