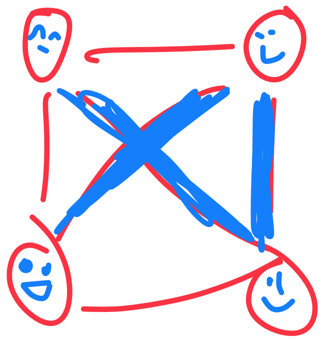
Data Structures
MST 2

CS 225
Brad Solomon

November 3, 2025



UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN
Department of Computer Science

Learning Objectives

Review the minimum spanning tree (with weights)

Review Kruskal's / Prim's MST Algorithms

Focus on determining Big O of complex pseudocode

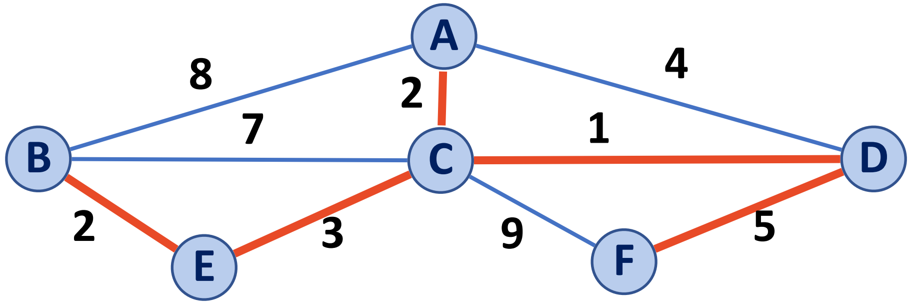
Compare implementations under different conditions

Minimum Spanning Tree Algorithms

Input: Connected, undirected graph **G** with edge weights (unconstrained, but must be additive)

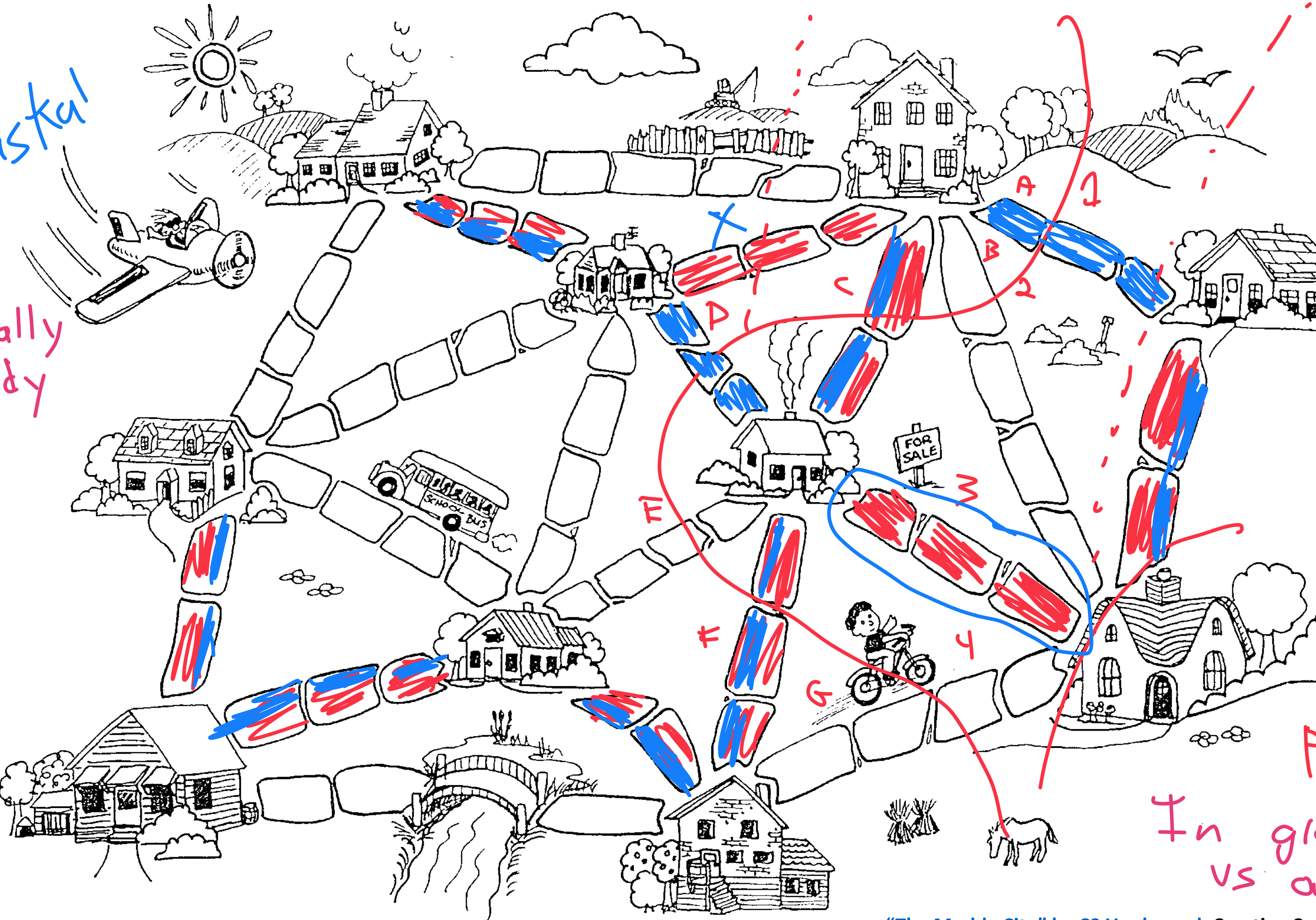
Output: A graph G' with the following properties:

- G' is a spanning graph of G
- G' is a tree (connected, acyclic)
- G' has a minimal total weight among all spanning trees



Klustal

Globally greedy

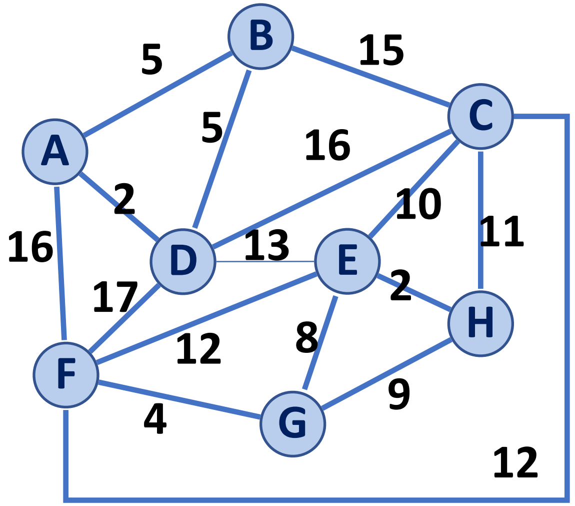


locally greedy

Prim

In group vs out group

Kruskal's Algorithm *(A graph algorithm for the MST problem)*



What information do I need to get efficiently?

1) The global minimum edge weight

3) If two vertices are already connected

Is edge useful

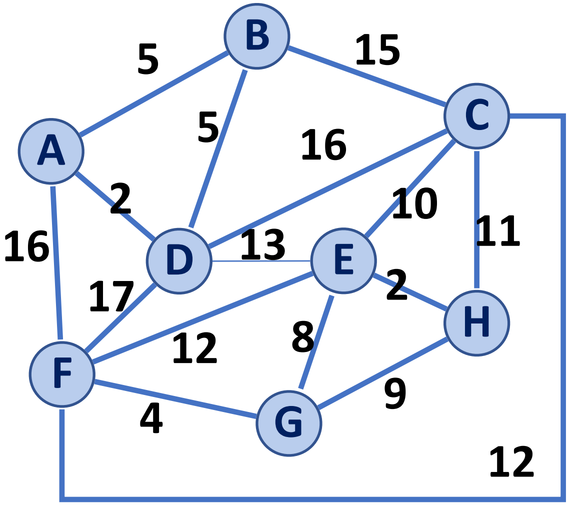
What does this tell me about my implementation choices?

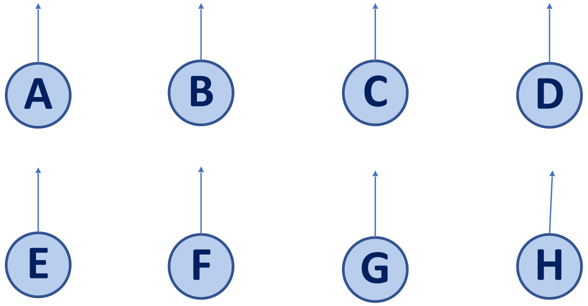
1) We need a **priority queue** of edges (sorted by weight)

2) We need a **disjoint set** of vertices

Kruskal's Algorithm

(A, D)
(E, H)
(F, G)
(A, B)
(B, D)
(G, E)
(G, H)
(E, C)
(C, H)
(E, F)
(F, C)
(D, E)
(B, C)
(C, D)
(A, F)
(D, F)





1) Build a **priority queue** on edges

↳ A min heap

or

↳ A sorted array

2) Build a **disjoint set** on vertices

↳ Every vertex is own set

3) walk through queue

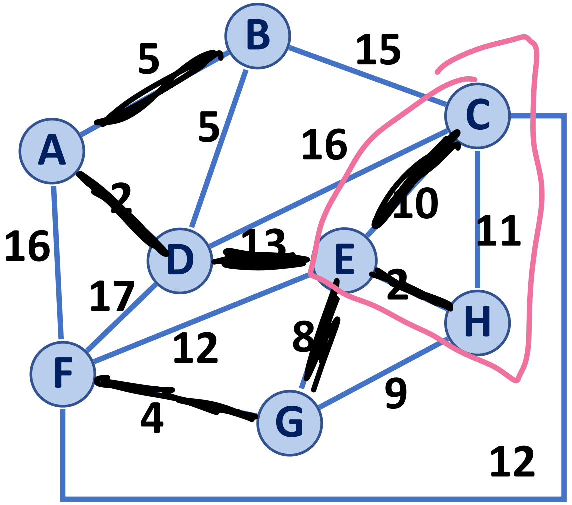
↳ pick min edge

↳ Add edge if connects two items

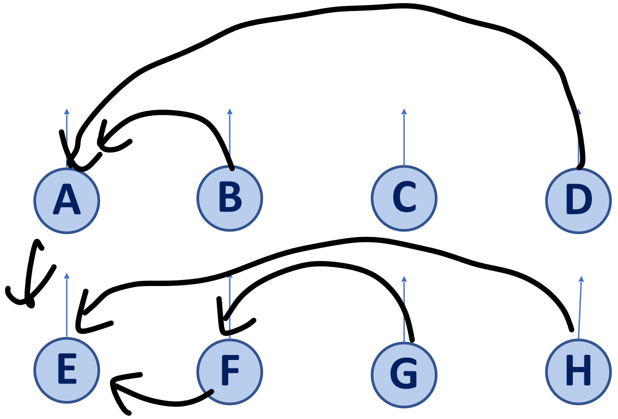
↳ updating disjoint set

Kruskal's Algorithm

(A, D)	✓
(E, H)	✓
(F, G)	✓
(A, B)	✓
(B, D)	✗
(G, E)	✓
(G, H)	✗
(E, C)	✓
(C, H)	✗
(E, F)	✗
(F, C)	✗
(D, E)	✓
(B, C)	
(C, D)	
(A, F)	
(D, F)	



At time step here



1) Build a **priority queue** on edges

A minheap

or

A sorted array

2) Build a **disjoint set** on vertices

All vertices start as their own set

3) Loop through min edges

If edge connects two disjoint sets

update *Union sets and record edge in MST*

4) Stop when:

N-1 edges recorded we have MST

Only a single disjoint set remains

Kruskal's Algorithm

```
KruskalMST(G):
  DisjointSets forest
  foreach (Vertex v : G.vertices()):      (2)
    forest.makeSet(v)

  PriorityQueue Q    // min edge weight   (1)
  Q.buildFromGraph(G.edges())

  Graph T = (V, {})
                                           3
  while |T.edges()| < n-1:
    Vertex (u, v) = Q.removeMin()
    if forest.find(u) != forest.find(v):
      T.addEdge(u, v)
      forest.union( forest.find(u),
                    forest.find(v) )
  if vertex u & v not in same set

  return T
```

1) Build a **priority queue** on edges

A minheap

or

A sorted array

2) Build a **disjoint set** on vertices

All vertices start as their own set

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Union sets and record edge in MST

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       forest.union( forest.find(u),
                     forest.find(v) )

  return T
```

$|V| = n, |E| = m$

What is the Big O?

$O(m)$

$O(m \log m)$

vs

$O(n \log m)$

Kruskal's Algorithm

|V|= n, |E|= m

```
1   KruskalMST(G):
2     DisjointSets forest
3     foreach (Vertex v : G.vertices()):
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6     PriorityQueue Q    // min edge weight
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9     Graph T = (V, {})
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11    while |T.edges()| < n-1:
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15        forest.union( forest.find(u),
16                      forest.find(v) )
17
18    return T
19
```

What is the Big O?

2 — 4: O(n)

6 — 7: Heap: O(m)
Sorted List: O(m log m)

11: m x <12-17>

12—17: Heap: O(log m)
Sorted List: O(1)

Disjoint set we treat as O(1) b/c path compression w/ smart union

Kruskal's Algorithm

Priority Queue:	Heap	Sorted Array
Building :7	$O(m)$	$O(m \log m)$
Each removeMin :12 $m \times$	$O(\log m)$	$O(1)$

Both are $O(m + m \log m)$

Why heap?

↳ In practice loop @ 11 less than m

↳ Better for dynamic datasets

Why sorted list?

↳ It produces a sorted list and doesn't destroy it!

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Kruskal's Algorithm

Priority Queue:	Heap	Sorted Array
Building :7	O(m)	O(m log m)
Each removeMin :12	O(m log m)	O(m)

Both result in m + m log m

Why is heap good?

If edge weights can change!

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18    return T
19
```

Why is sorted array good?

Sorted array not destroyed and can be useful in other algorithms!

Kruskal's Algorithm

connected $\qquad$ complete

$\underline{n-1} \le \underline{m} \le \underline{\sim n^2}$

edges $\qquad$ vertices

Priority Queue:	Total Running Time
Heap	$O(n) + O(m) + O(m \log m)$
Sorted Array	$O(n) + O(m \log m) + O(m)$

$O(n) + O(m \log m)$

Trivia fact / theory lesson:

$O(\log m) \approx O(\log n)$?

$\hookrightarrow O(\log n^2) = O(2 \log n)$

$= O(\log n)$

```
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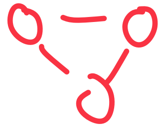
```

$O(n)$



$2 + 1 = 3$

Kruskal's Algorithm

Priority Queue:	Total Running Time
Heap	O(n)+O(m) + O(m log m)
Sorted Array	O(n) + O(m log m) + O(m)

```
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                     forest.find(v) )

  return T
```

Neat trick: O(log m) = O(log n)

Bounds of n & m relationship?

$$n - 1 \leq m \leq n^2$$

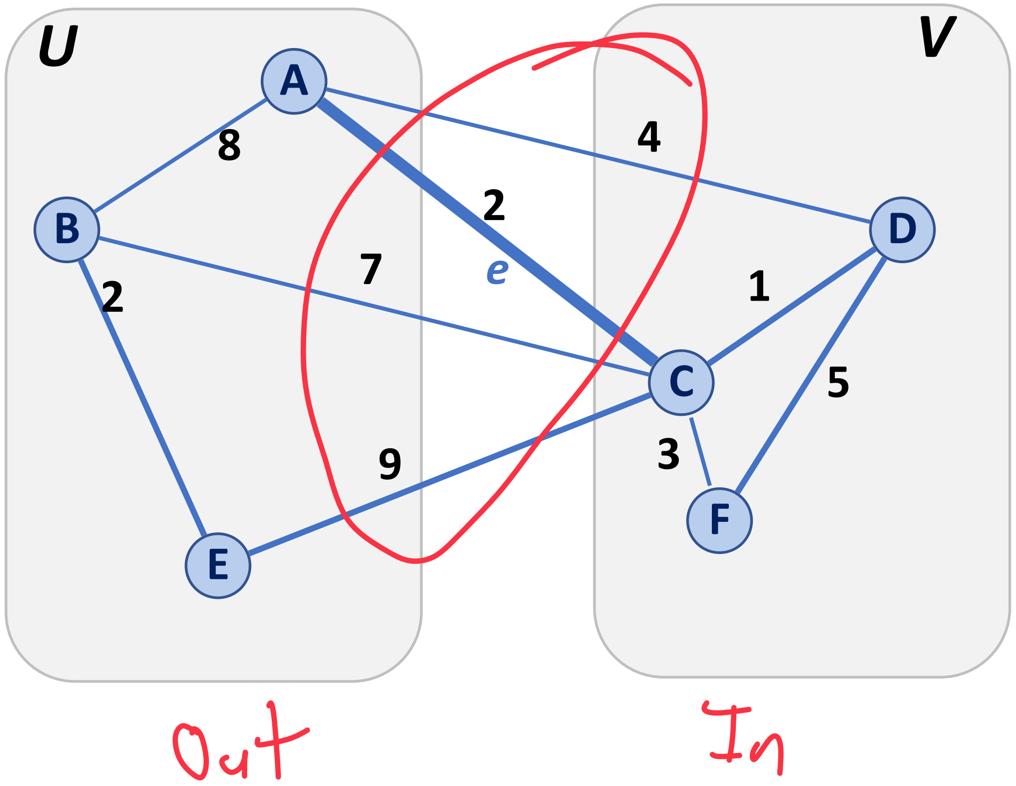
Connected graph *Complete graph*

Partition Property

Consider an arbitrary partition of the vertices on **G** into two subsets **U** and **V**.

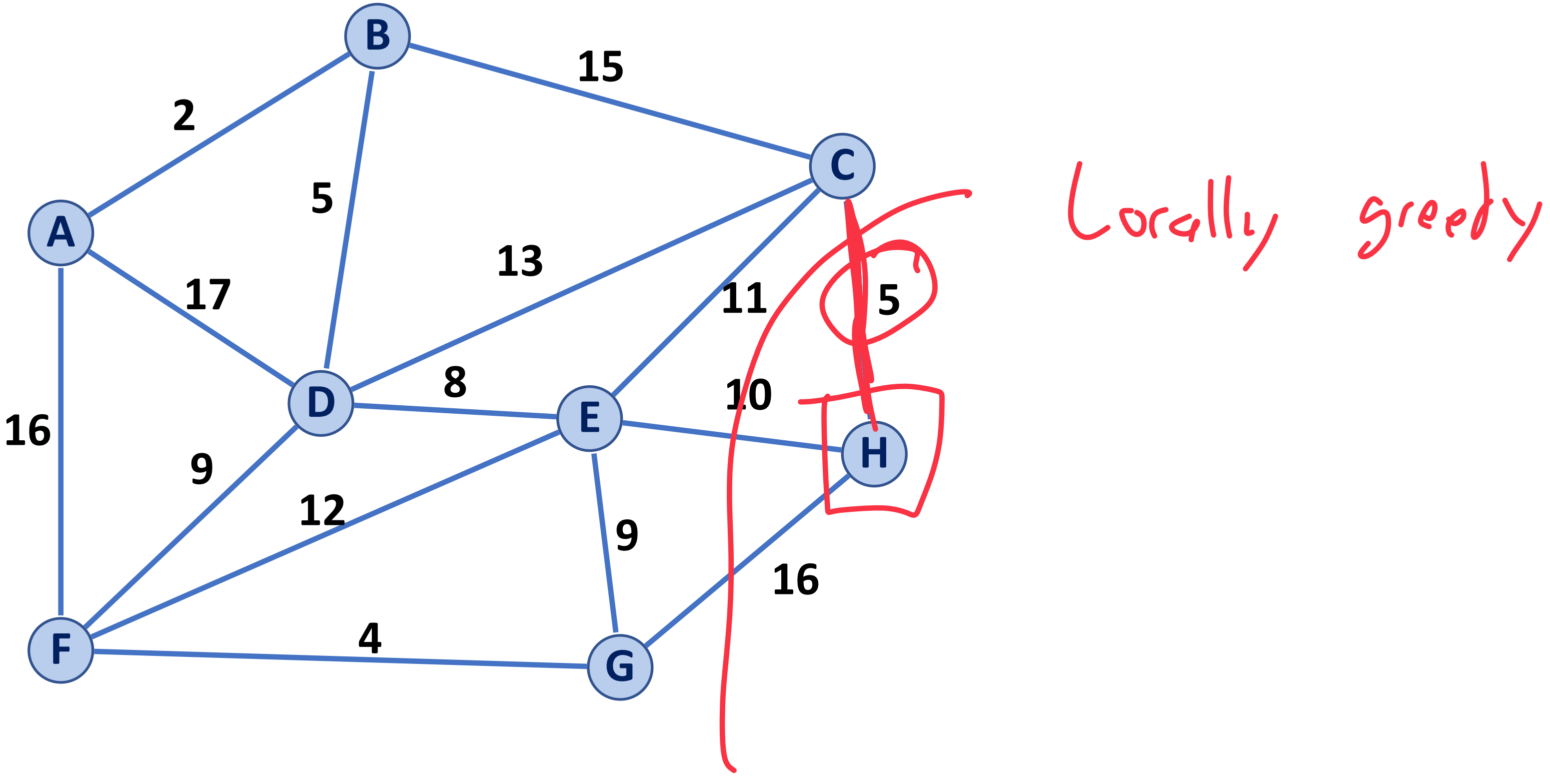
Let **e** be an edge of minimum weight across the partition.

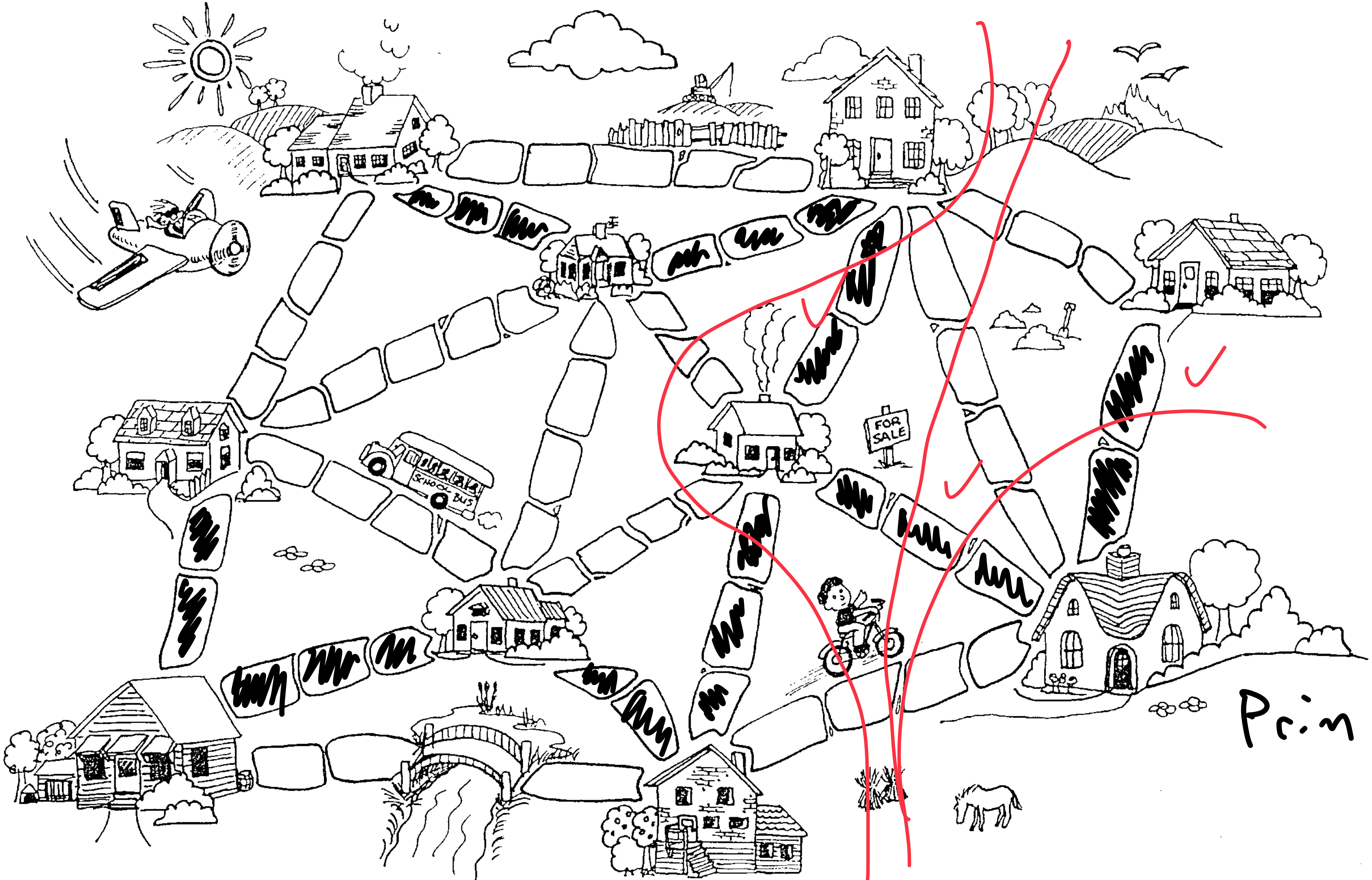
Then **e** is part of some minimum spanning tree.



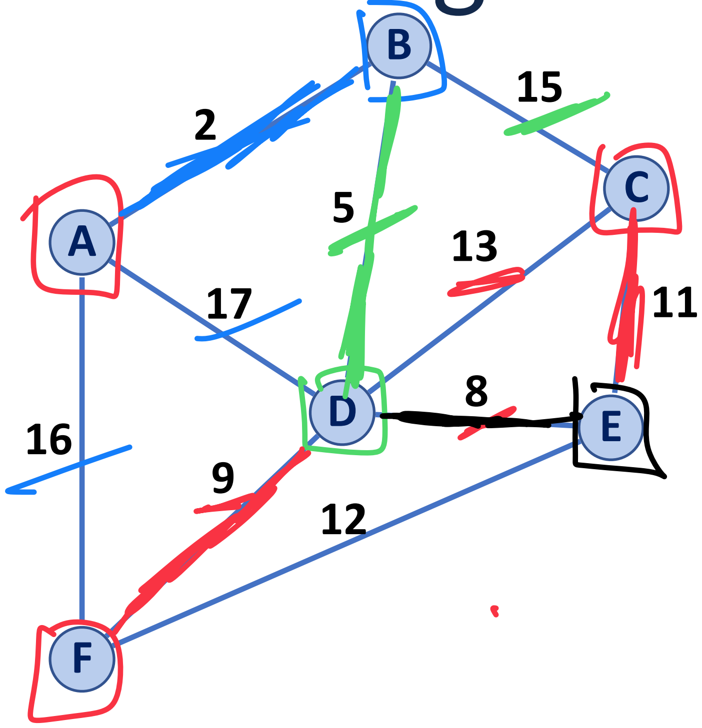
Partition Property

The partition property suggests an algorithm:




SCHOOL BUS
FOR SALE
Prim

Prim's Algorithm



A	B	C	D	E	F
0, -	~~∞~~ 2, A	~~∞~~ ~~15, B~~ ~~13, D~~ 11, F	~~∞~~ ~~17, A~~ 5, B	~~∞~~ 8, D	~~∞~~ ~~16, A~~ 9, D

```
PrimMST(G, s):
  Input: G, Graph;
         s, vertex in G, starting vertex
  Output: T, a minimum spanning tree (MST) of G

  foreach (Vertex v : G.vertices()):
    d[v] = +inf
    p[v] = NULL
  d[s] = 0

  PriorityQueue Q   // min distance, defined by d[v]
  Q.buildHeap(G.vertices())
  Graph T           // "labeled set"

  repeat n times:
    Vertex m = Q.removeMin()
    T.add(m)
    foreach (Vertex v : neighbors of m not in T):
      if cost(v, m) < d[v]:
        d[v] = cost(v, m)
        p[v] = m

  return T
```

Init

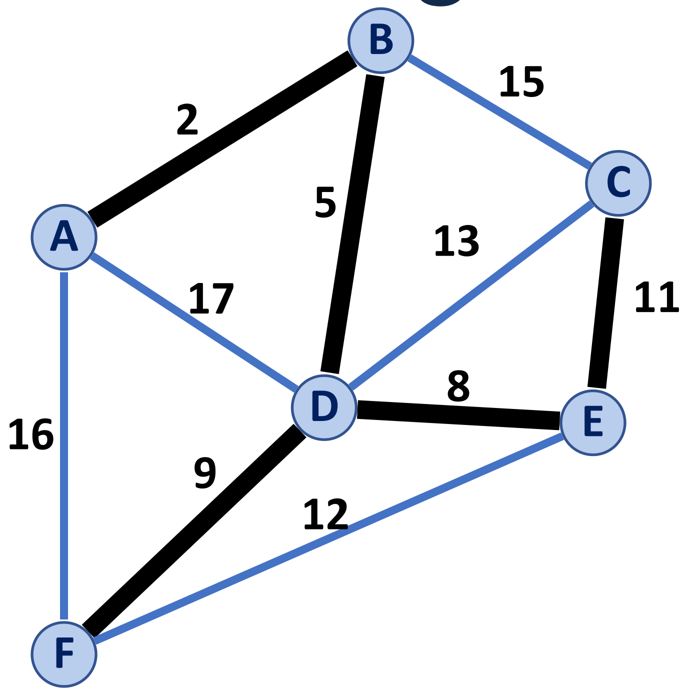
← All distances are ∞

starts off removing A

update cost/distance if new edge smaller!

update prev so we know source

Prim's Algorithm



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  return T
```

I pick smallest distance

Prim's Big O

|V|= n, |E|= m

```
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```

O(n)

Init

minheap: O(n)

Unsorted array: O(1)

Prim's Big O

|V|= n, |E|= m

7 — 9: O(n)

12—14:

MinHeap: O(n)

Unsorted Array: O(1)

16—22: Complicated!

```
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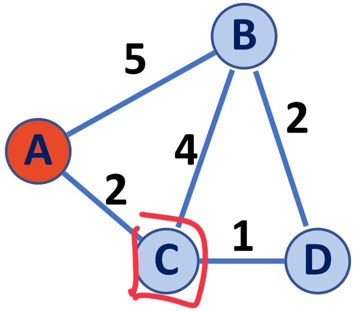
21          d[v] = cost(v, m)
22          p[v] = m
23
```

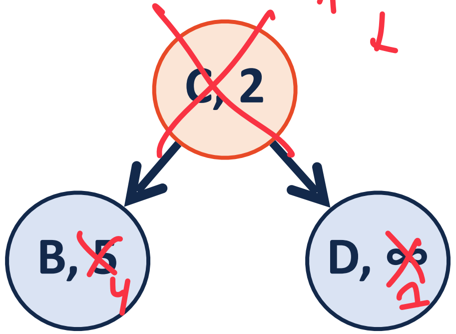
Depends on choice of **PriorityQueue** (MinHeap vs Unsorted Array)

Depends on choice of **Graph** (Adjacency Matrix vs Adjacency List)

A	B	C	D
0	~~5~~ 4	2	~~∞~~ 1



heap ↑ ↓



D,1
B,4

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        d[v] = cost(v, m)
        p[v] = m
```

O(n)

graph has this

pretend O(1) for unordered array

n x

Adj Matrix

n

n

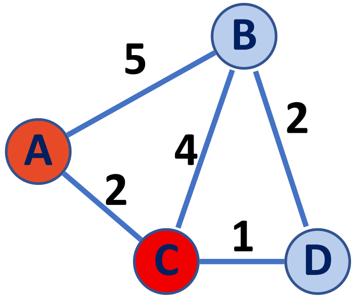
O(n) for one vertex

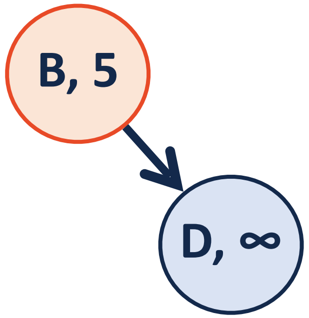
n x O(deg(v))

$\sum_v deg(v) = 2m$

	Adj. Matrix	Adj. List
Heap	O(n) + ____________ + O(n^2) + __________________	O(n) + ____________ + O(m) + __________________

A	B	C	D
0	5	2, A	∞



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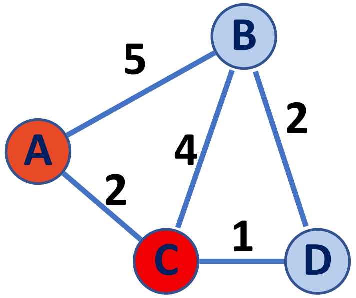
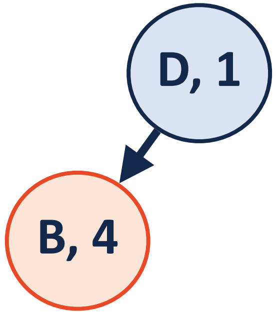
```

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + ________________	O(n) + O(n log n) + O(m) + ________________

A	B	C	D
0	4	2, A	1



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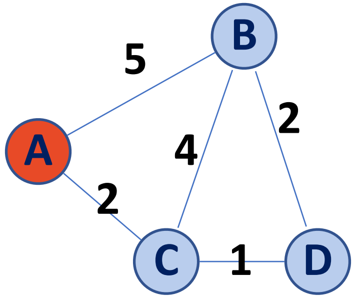
        d[v] = cost(v, m)
        p[v] = m
```

1) Change minheap value

2) HeapifyUp()

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + O(m log n)	O(n) + O(n log n) + O(m) + O(m log n)

(A, 0)
(D, ∞)
(C, 2)
(B, 5)

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```

	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$
Unsorted Array		

Prim's Algorithm

Sparse Graph:

Dense Graph:

```
PrimMST(G, s):
  foreach (Vertex v : G.vertices()):
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        p[v] = m

```

	Adj. Matrix	Adj. List
Heap	$O(n^2 + m\lg(n))$	$O(n\lg(n) + m\lg(n))$
Unsorted Array	$O(n^2)$	$O(n^2)$

MST Algorithm Runtime:

Kruskal's Algorithm:

O(n + m log (n))

Prim's Algorithm:

O(n log(n) + m log (n))

Sparse Graph:

Dense Graph:

Suppose I have a new heap:

	Binary Heap	Fibonacci Heap
Remove Min	O(lg(n))	O(lg(n))
Decrease Key	O(lg(n))	O(1)*

What's Prim's updated running time?

```
PrimMST(G, s):
  foreach (Vertex v : G.vertices()):
    d[v] = +inf
    p[v] = NULL
  d[s] = 0

  PriorityQueue Q // min distance, defined by d[v]
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```

Shortest Path