Data Structures and Algorithms

Bloom Filters 3 & Cardinality Intro

CS 225

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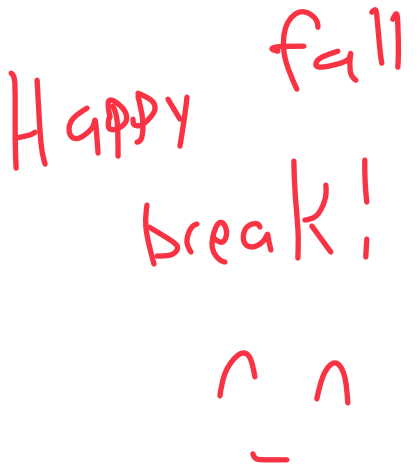
Department of Computer Science



Learning Objectives

Finish discussing bloom filters (and review bit vectors)

Introduce the concept of cardinality and cardinality estimation

Applications

Bloom Filter: Error Rate

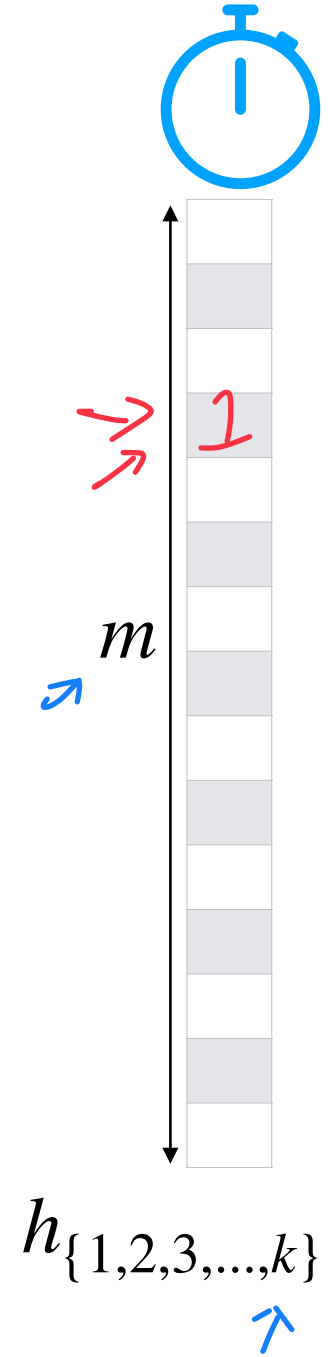
Given bit vector of size m and k SUHA hash function

What is our expected FPR after n objects are inserted?

The probability my bit is 1 after n objects inserted

$$\left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^{k}$$

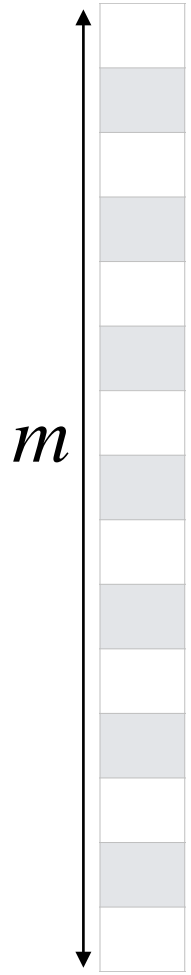
The number of [assumed independent] trials



Bloom Filter: Error Rate

$h_{\{1,2,3,\ldots,k\}}$

Vector of size m, k SUHA hash function, and n objects



To minimize the FPR, do we prefer…

(A) large k **(B) small** k

$$\left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^{k}$$

Bloom Filter: Error Rate

Vector of size m, k SUHA hash function, and n objects

(A) large k

$$\left(1-\left(1-\frac{1}{m}\right)^{nk}\right)^{k}$$

As k increases, this gets smaller!

↓ FPR approaches 1

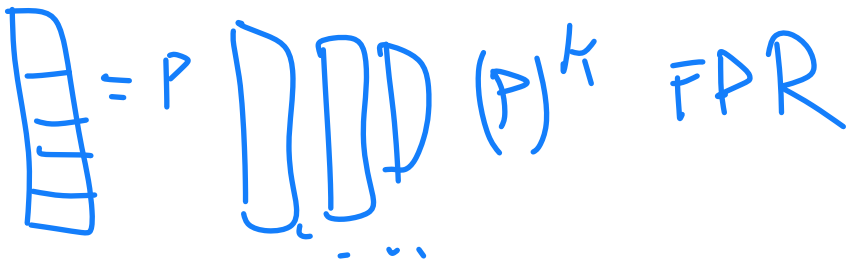
≈ 100% FPR

(B) small k

$$\left(1-\left(1-\frac{1}{m}\right)^{nk}\right)^{k}$$

As k decreases, this gets smaller!

↳ Repeated random trials



Bloom Filter: Optimal Error Rate

k = hash
M = BF size
n = items

To build the optimal hash function, fix **m** and **n**!

Claim: The optimal hash function is when $k^* = \ln 2 \cdot \frac{m}{n}$ ↩ $\alpha = n/m$

(1) $\left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^k \approx \left(1 - e^{\frac{-nk}{m}}\right)^k$

Shape of eq more clear

(2) $\frac{d}{dk}\left(1 - e^{\frac{-nk}{m}}\right)^k \approx \frac{d}{dk}\left(k \ \ln(1 - e^{\frac{-nk}{m}})\right)$

Bloom Filter: Optimal Error Rate

Claim 1: $\left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^{k} \approx \left(1 - e^{\frac{-nk}{m}}\right)^{k}$

$$\left(1 - \frac{1}{m}\right)^{nk} = e^{ln\left[\left(1 - \frac{1}{m}\right)^{nk}\right]}$$

$X = e^{\ln(x)^{Y}} = e^{Y \ln(x)}$

Bloom Filter: Optimal Error Rate

Claim 1: $\left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^{k} \approx \left(1 - e^{\frac{-nk}{m}}\right)^{k}$

$$\left(1 - \frac{1}{m}\right)^{nk} = e^{ln\left[\left(1 - \frac{1}{m}\right)^{nk}\right]}$$

$$= e^{ln\left[\left(1 - \frac{1}{m}\right)\right]nk}$$

Bloom Filter: Optimal Error Rate

Taylors expansion of $ln(1+x)$:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

"Mercator Series"

$e^{\ln\left[\left(1-\frac{1}{m}\right)\right]nk}$

$X = -\frac{1}{m}$

This is Small

$$\left(1 - \frac{1}{m}\right)^{nk} \approx e^{\frac{-nk}{m}}$$

Bloom Filter: Optimal Error Rate

Claim 1: $\left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^{k} \approx \left(1 - e^{\frac{-nk}{m}}\right)^{k}$

$$\left(1 - \frac{1}{m}\right)^{nk} = e^{ln\left[\left(1 - \frac{1}{m}\right)^{nk}\right]}$$

$$= e^{ln\left[\left(1 - \frac{1}{m}\right)\right]nk}$$

$$\approx e^{\frac{-nk}{m}}$$

Bloom Filter: Optimal Error Rate

Claim 2: $$\frac{d}{dk}\left(1 - e^{\frac{-nk}{m}}\right)^k \approx \frac{d}{dk}\left(k \ \ ln(1 - e^{\frac{-nk}{m}})\right)$$

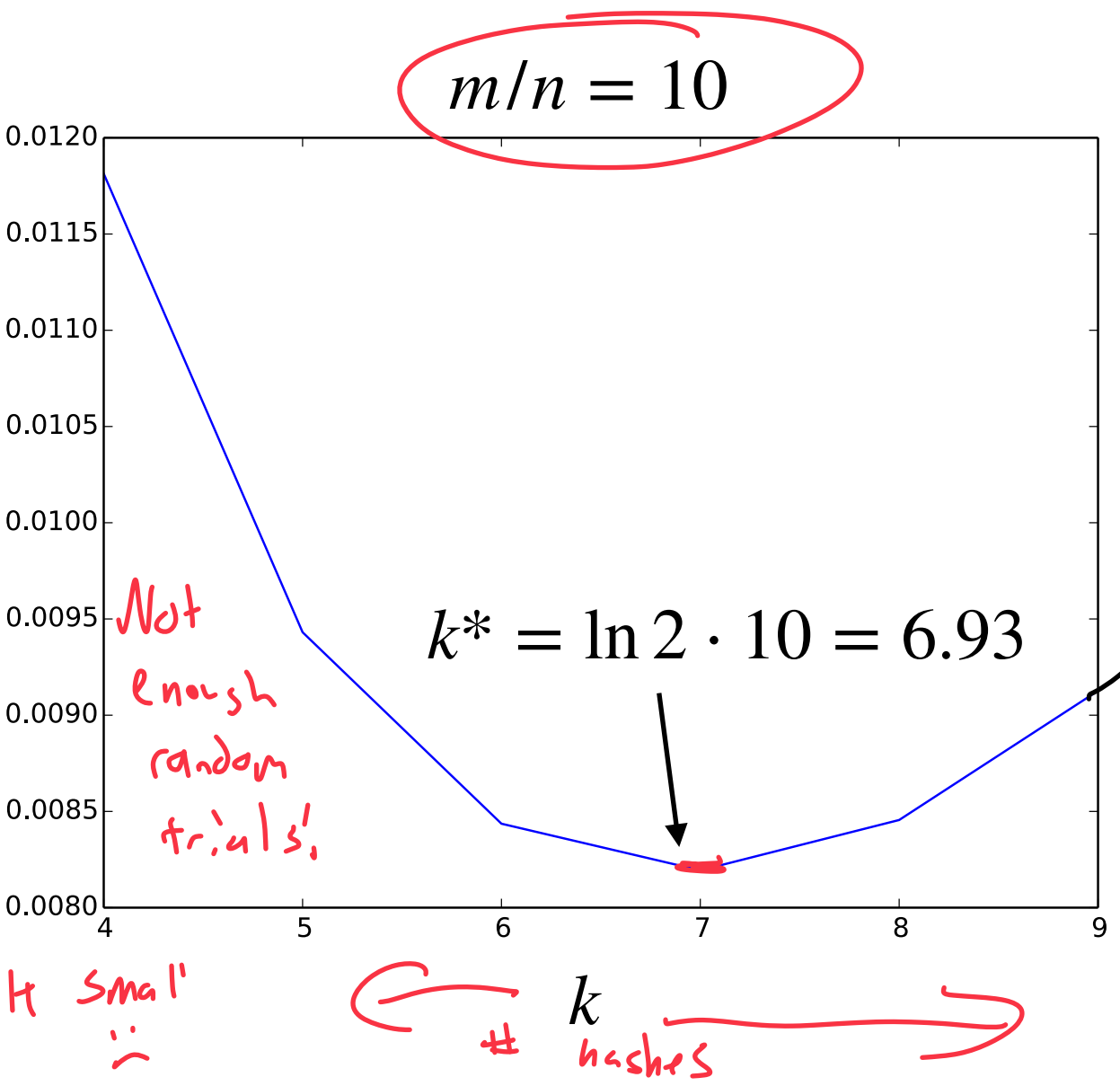
Fact: $$\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} \frac{df(x)}{dx}$$

TL;DR: $min\left[f(x)\right] = min\left[\ln f(x)\right]$

Take deriv of ln

Derivative is zero when $k^* = \ln 2 \cdot \frac{m}{n}$

Bloom Filter: Error Rate



FPR $\left(1 - e^{\frac{-nk}{m}}\right)^k$

BF becomes too saturated w/ 1s

Figure by Ben Langmead

Bloom Filter: Optimal Parameters

$$k^* = \ln 2 \cdot \frac{m}{n}$$

Given any two values, we can optimize the third

$3 = \ln 2 \left(\frac{m}{100}\right)$

$n = 100$ items $\quad k = 3$ hashes $\quad m = \frac{300}{\ln 2} = \sim 433$ bits

$m = 100$ bits $\quad n = 20$ items $\quad k = \ln(2)\ \frac{100}{20} \sim 3.5$ hashes

3 ↙ ↓ 4

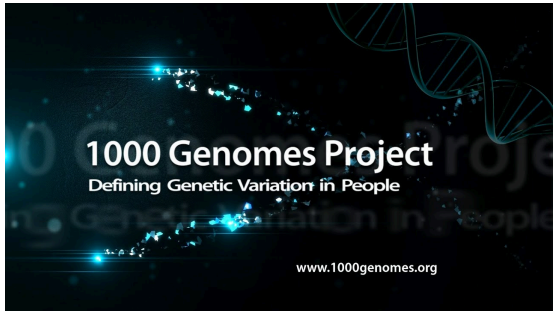
$m = 100$ bits $\quad k = 2$ items $\quad n = \ln(2)\ \frac{100}{2} \sim 35$ items

Bloom Filter: Optimal Parameters

$$m = \frac{nk}{\ln 2} \approx 1.44 \cdot nk$$

Optimal hash function is still O(n)!

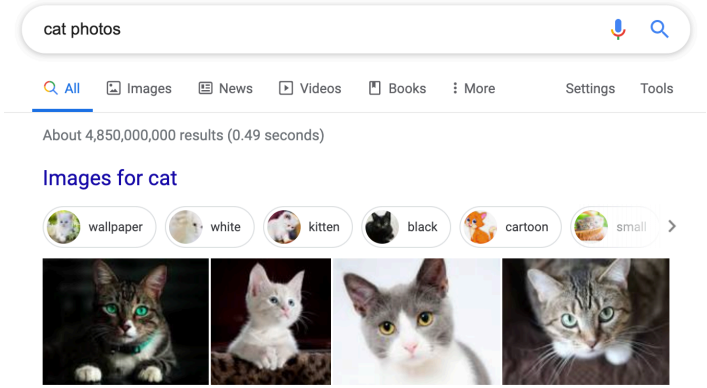
↳ Don't need to be optimal!



n = 250,000 files vs ~10^{15} nucleotides vs 260 TB

(characters T, C, G, A)

↳ 1.44 bits × 250,000 = 45 KB w/ 1 hash
90 KB 2 hashes



n = 60 billion — 130 trillion

Bloom Filters

A probabilistic data structure storing a set of values

$h_{\{1,2,3,\ldots,k\}}$

Has three key properties:

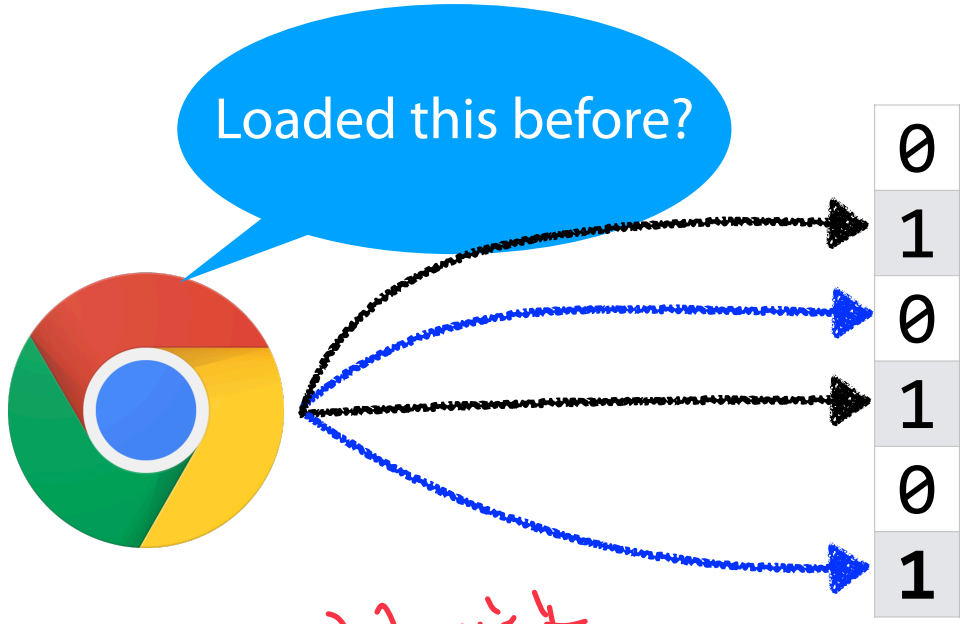
- k, number of hash functions
- n, expected number of insertions
- m, filter size in bits

Expected false positive rate: $\left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^{k} \approx \left(1 - e^{\frac{-nk}{m}}\right)^{k}$

Optimal accuracy when: $k^* = \ln 2 \cdot \frac{m}{n}$

Bloom Filter: Website Caching

False Positive is not a problem!



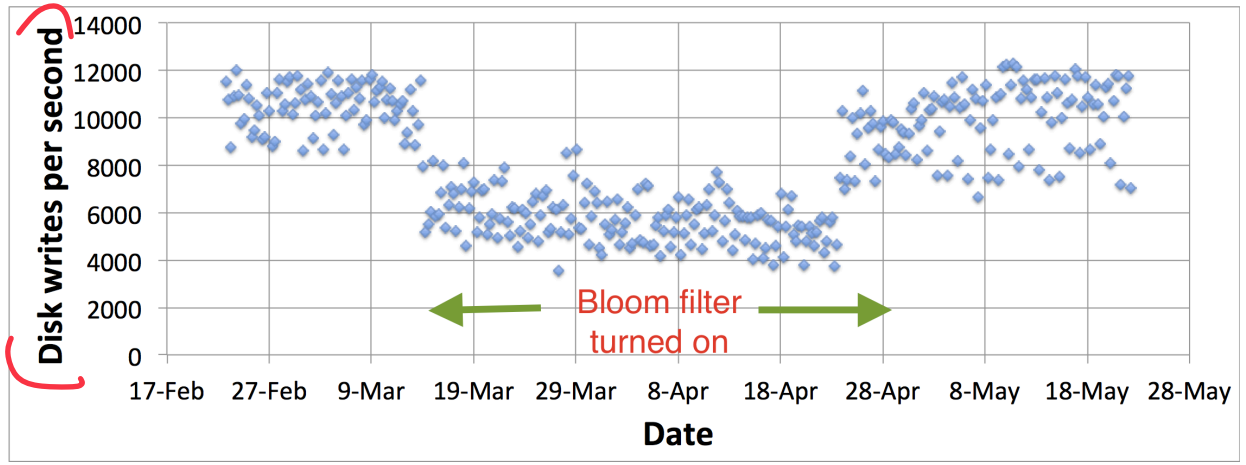
Cache this page!

↳ Caching a few extra pages isn't bad

Add to filter (but don't cache!)

↳ Cache on 2nd visit

↳ Sometimes on first



Maggs, Bruce M., and Ramesh K. Sitaraman. **Algorithmic nuggets in content delivery.** *ACM SIGCOMM Computer Communication Review* 45.3 (2015): 52-66.

Bitwise Operators in C++

Array of n bits

How can we encode a bit vector in C++?

vector<bool>

None are particularly good

uint_8 - 8 bits

"

char - 8 bits

work w/ extra steps!

↳ Bit operations fix

Trivia - vector<bool> doesn't make vector

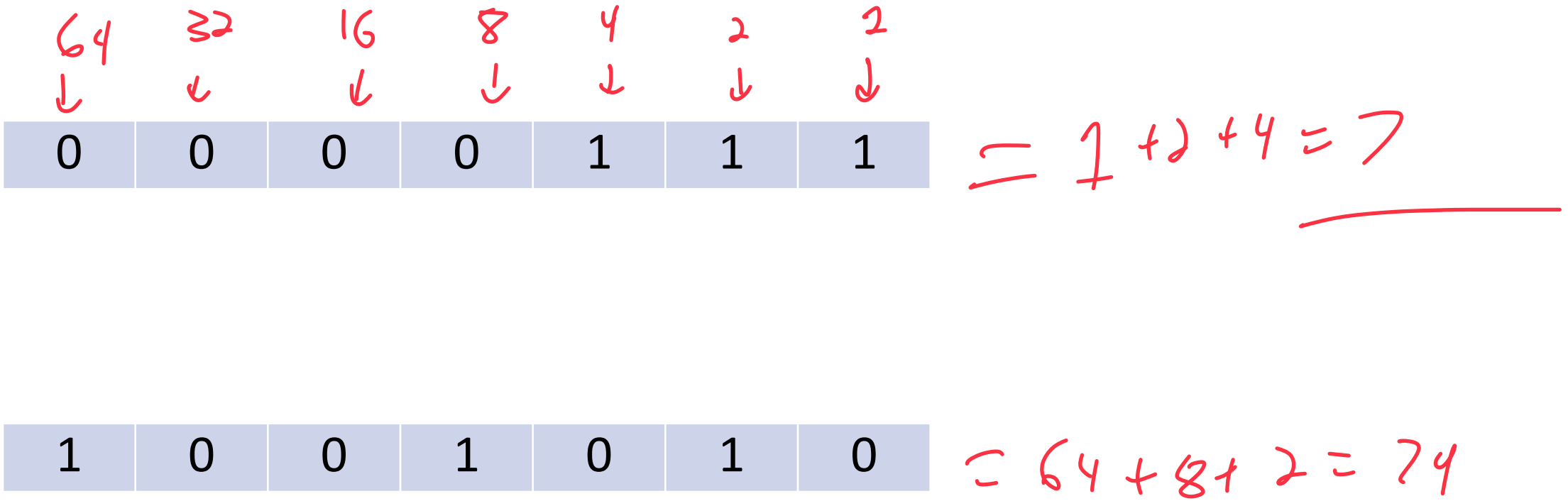
↳ can't be multi threaded ↳ Doesn't implement parts of vector

	0	1	2	3	4	5	6	7	← Look up entire char
....	1	0	1	0	1	0	1	0	....

Bitwise Operators in C++

Traditionally, bit vectors are read from RIGHT to LEFT

Warning: Lab_Bloom won't do this



Bitwise Operators in C++

Let **A = 10110** (Most ... Least) Let **B = 01110**

~B: 10001 (invert all 1s and 0s)

A & B: 00110 (Both have to be 1 for 1, else 0)

B & 00100 → 1 if 3 is 1, 0 otherwise

A | B: 11110 (If either is 1, set 1, else 0)

A >> 2: 00101 (discard least sig bits and replace most w 0)

B << 2: 11000 (discard most sig bits & replace least w 0)

Bit Vectors: Unioning

Bit Vectors can be trivially merged using bit-wise union.

0	1		0	0		0	1
1	0		1	1		1	1
2	1		2	1		2	1
3	1		3	0		3	1
4	0	U	4	0	=	4	0
5	0		5	0		5	0
6	1		6	1		6	1
7	0		7	1		7	1
8	0		8	1		8	1
9	1		9	1		9	1

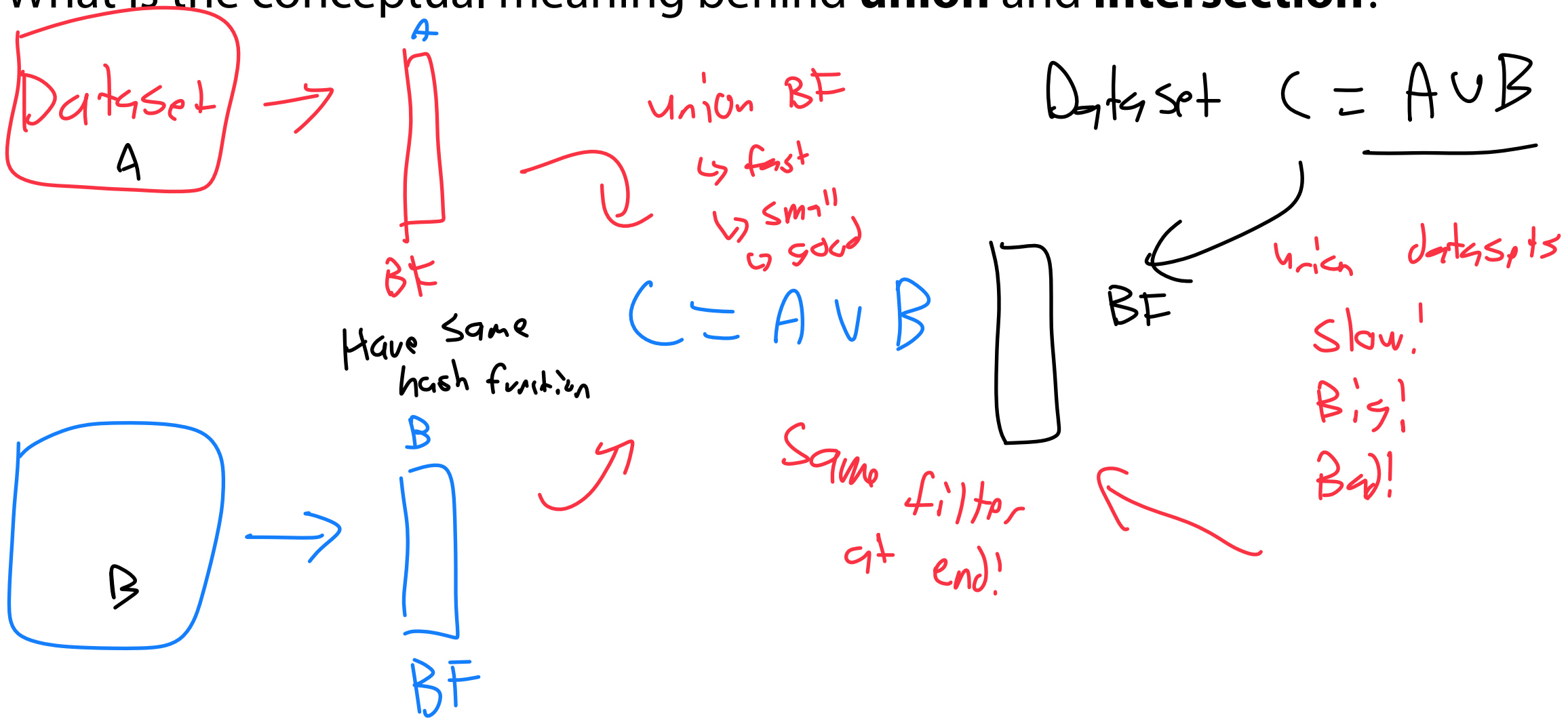
Bit Vectors: Intersection

Bit Vectors can be trivially merged using bit-wise intersection.

0	1		0	0		0	
1	0		1	1		1	
2	1		2	1		2	
3	1		3	0		3	
4	0	U	4	0	=	4	
5	0		5	0		5	
6	1		6	1		6	
7	0		7	1		7	
8	0		8	1		8	
9	1		9	1		9	

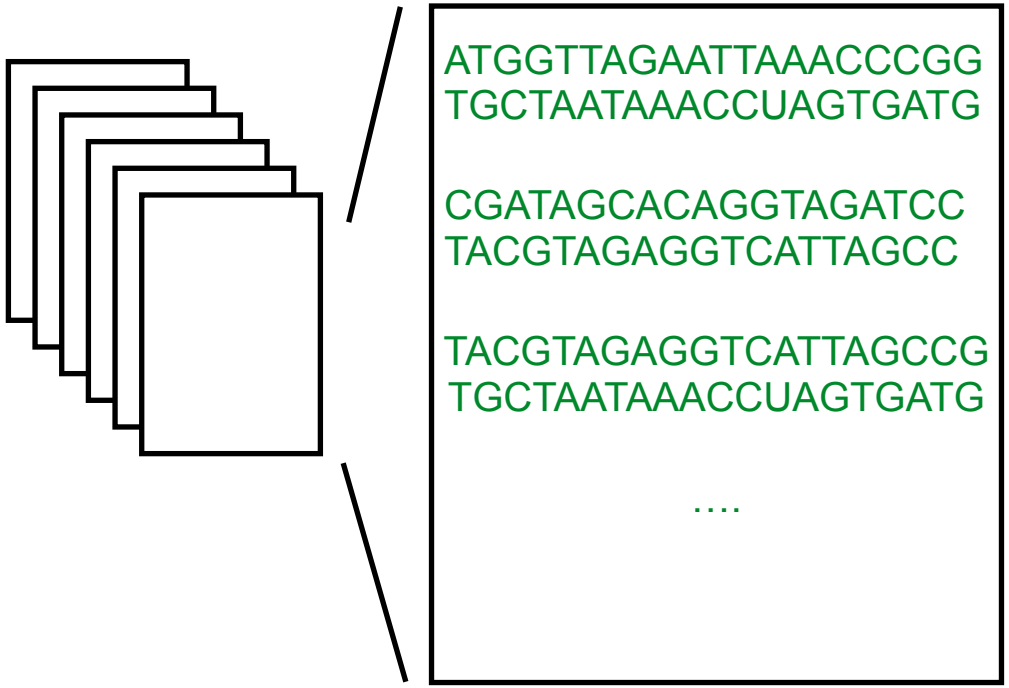
Bit Vector Merging

What is the conceptual meaning behind **union** and **intersection**?

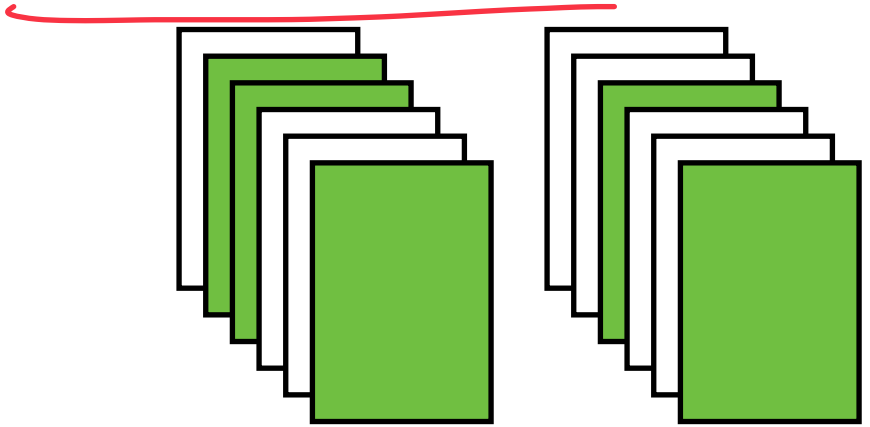


Sequence Bloom Trees

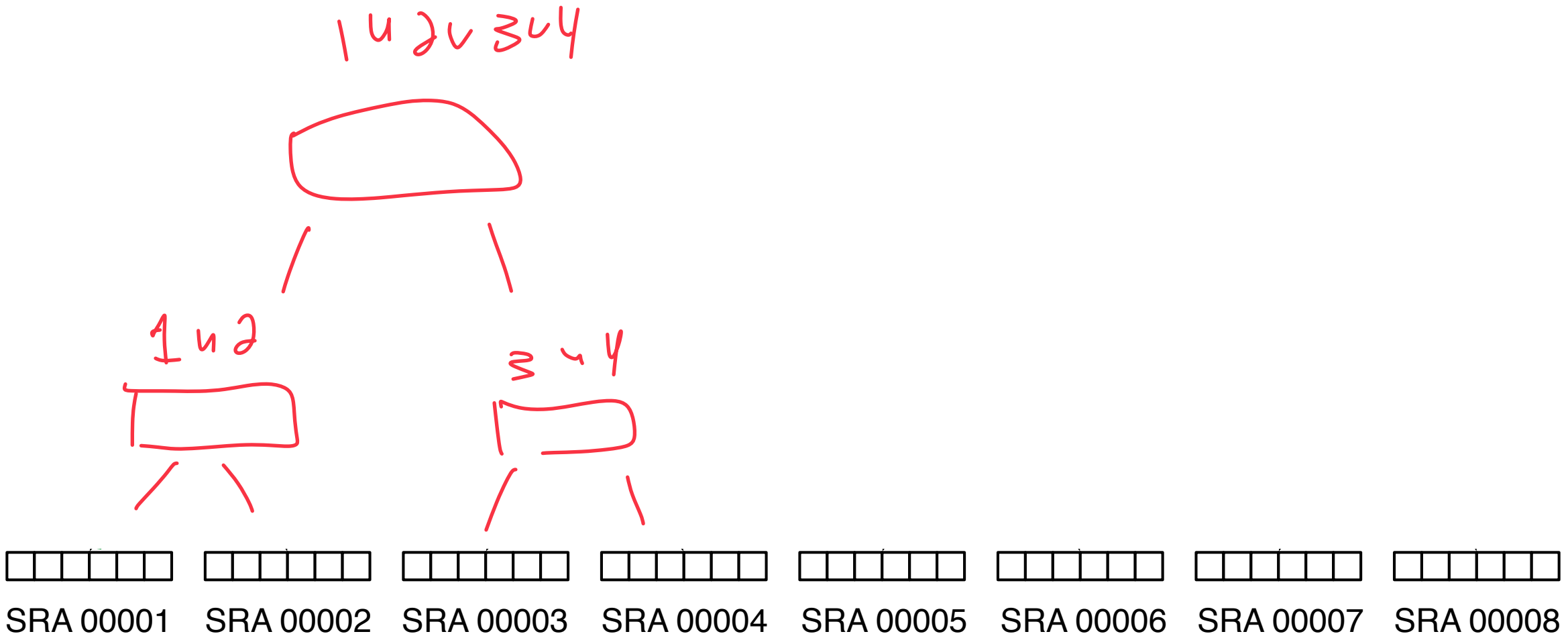
Imagine we have a large collection of text…



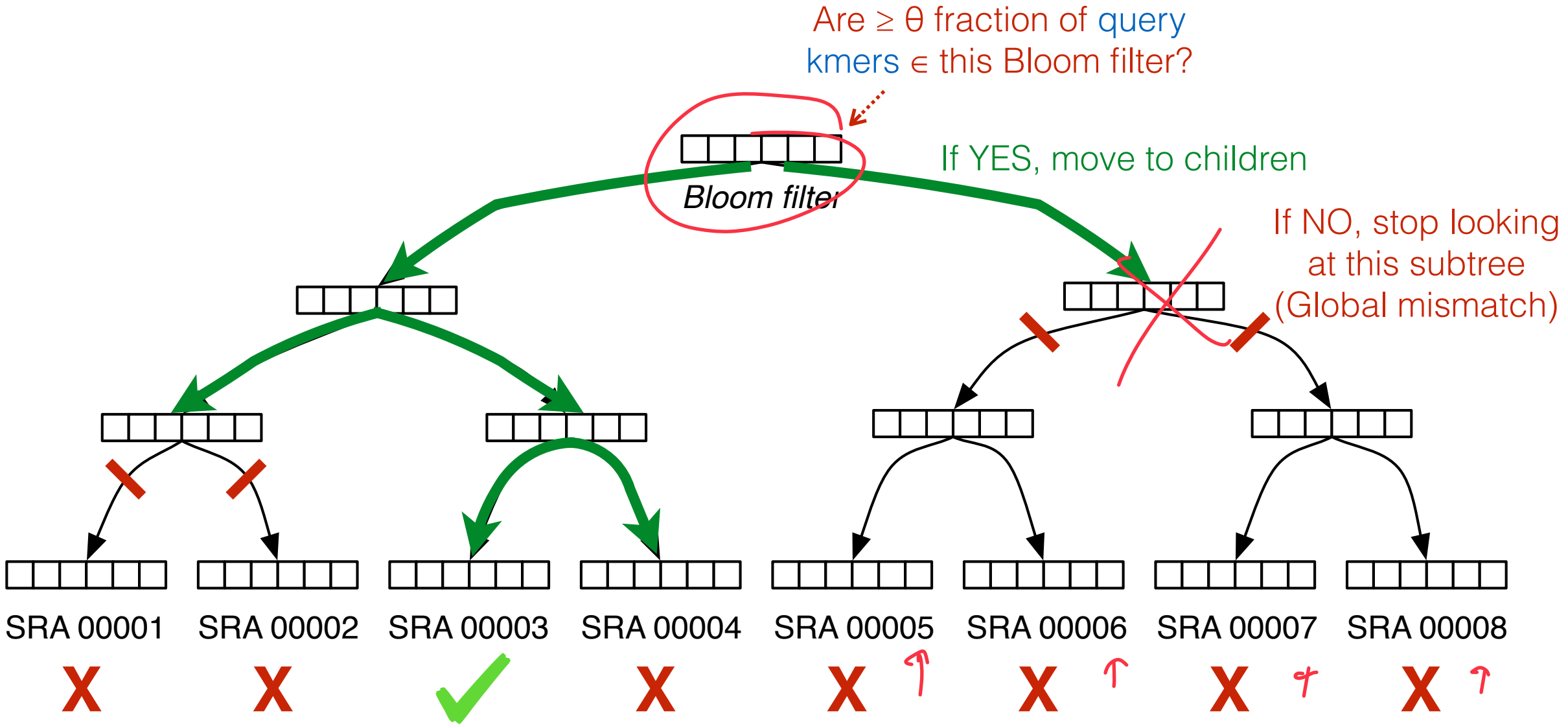
And our goal is to search these files for a query of interest…



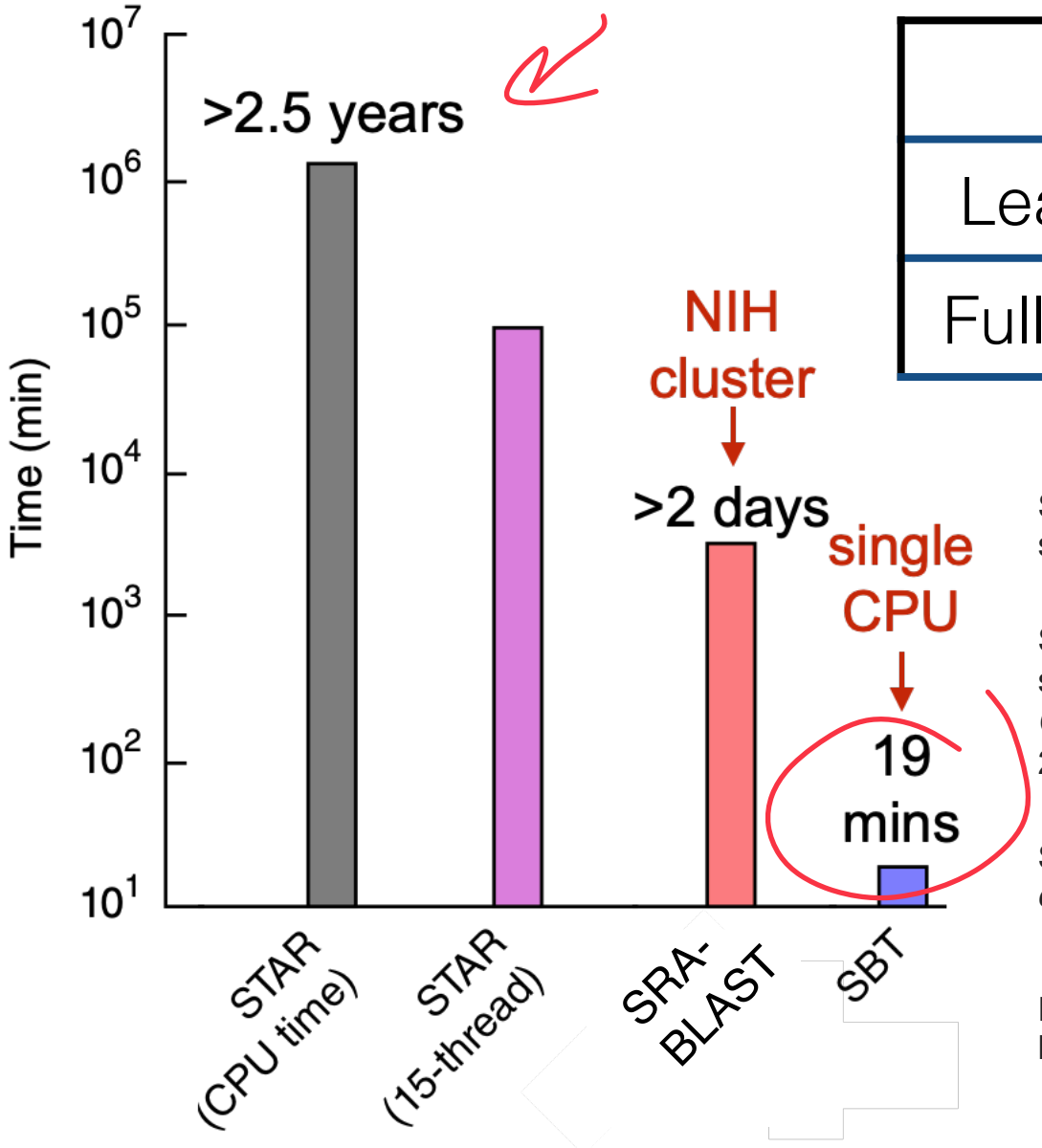
Sequence Bloom Trees



Sequence Bloom Trees



Sequence Bloom Trees



	SRA	FASTA.gz	SBT
Leaves	4966 GB	2692 GB	63 GB
Full Tree	-	-	200 GB

Solomon, Brad, and Carl Kingsford. "Fast search of thousands of short-read sequencing experiments." *Nature biotechnology* 34.3 (2016): 300-302.

Solomon, Brad, and Carl Kingsford. "Improved search of large transcriptomic sequencing databases using split sequence bloom trees." *International Conference on Research in Computational Molecular Biology*. Springer, Cham, 2017.

Sun, Chen, et al. "Allsome sequence bloom trees." *International Conference on Research in Computational Molecular Biology*. Springer, Cham, 2017.

Harris, Robert S., and Paul Medvedev. "Improved representation of sequence bloom trees." *Bioinformatics* 36.3 (2020): 721-727.

Bloom Filters: Tip of the Iceberg

Cohen, Saar, and Yossi Matias. "Spectral bloom filters." *Proceedings of the 2003 ACM SIGMOD international conference on Management of data*. 2003.

Fan, Bin, et al. "Cuckoo filter: Practically better than bloom." *Proceedings of the 10th ACM International on Conference on emerging Networking Experiments and Technologies*. 2014.

Nayak, Sabuzima, and Ripon Patgiri. "countBF: A General-purpose High Accuracy and Space Efficient Counting Bloom Filter." *2021 17th International Conference on Network and Service Management (CNSM)*. IEEE, 2021.

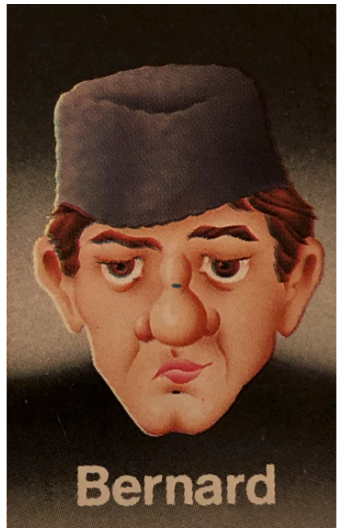
Mitzenmacher, Michael. "Compressed bloom filters." *IEEE/ACM transactions on networking* 10.5 (2002): 604-612.

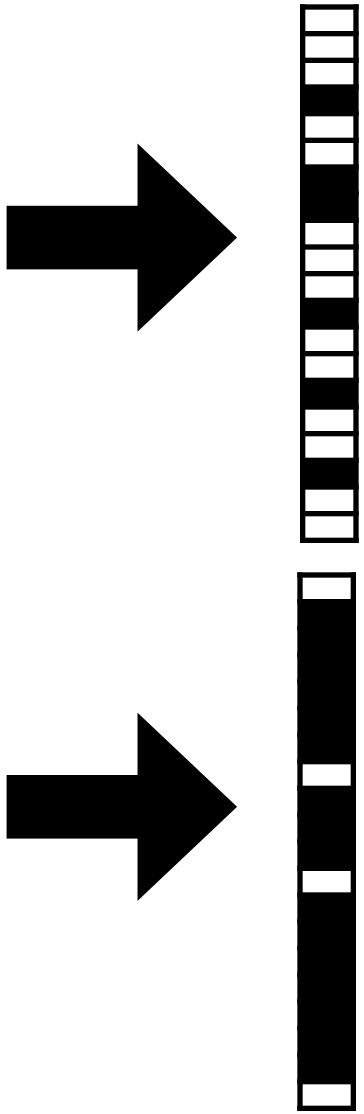
Crainiceanu, Adina, and Daniel Lemire. "Bloofi: Multidimensional bloom filters." *Information Systems* 54 (2015): 311-324.

Chazelle, Bernard, et al. "The bloomier filter: an efficient data structure for static support lookup tables." *Proceedings of the fifteenth annual ACM-SIAM symposium on Discrete algorithms*. 2004.

There are many more than shown here…

The hidden problem with (most) sketches… SYHA!





Super Saturated!

↳ Cancer → New strings not found in healthy people

Cardinality

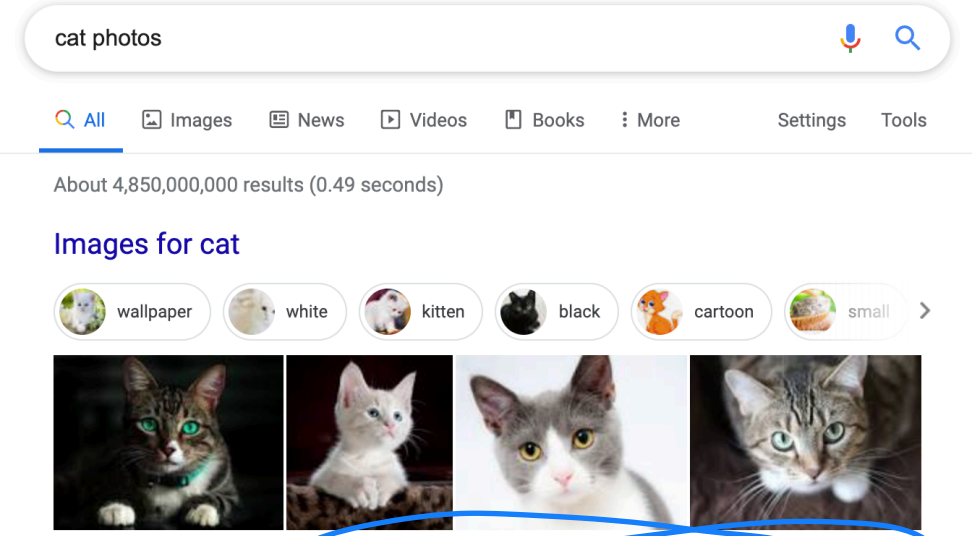
Cardinality is a measure of how many unique items are in a set

2
4
9
3
7
9
7
8
5
6

Cardinality

Sometimes its not possible or realistic to count all objects!

↳ Don't know data size!



Estimate: 60 billion — 130 trillion

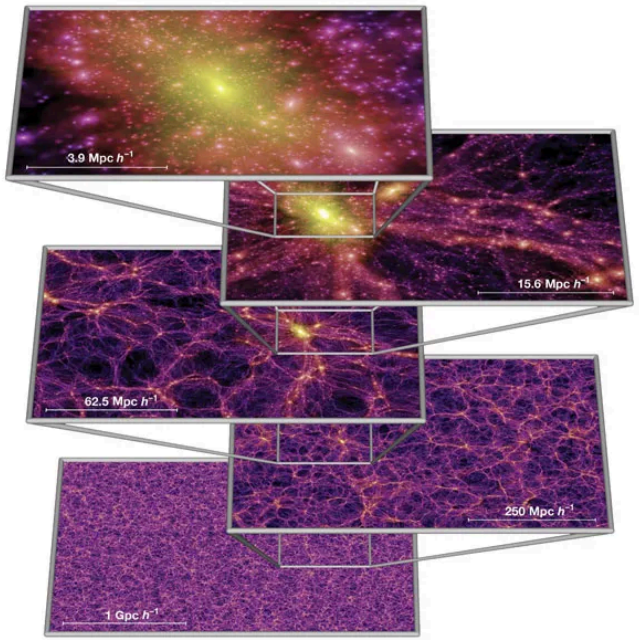


Image: https://doi.org/10.1038/nature03597

5581
8945
6145
8126
3887
8925
1246
8324
4549
9100
5598
8499
8970
3921
8575
4859
4960
42
6901
4336
9228
3317
399
6925
2660
2314

Cardinality Estimation

1 – 1000

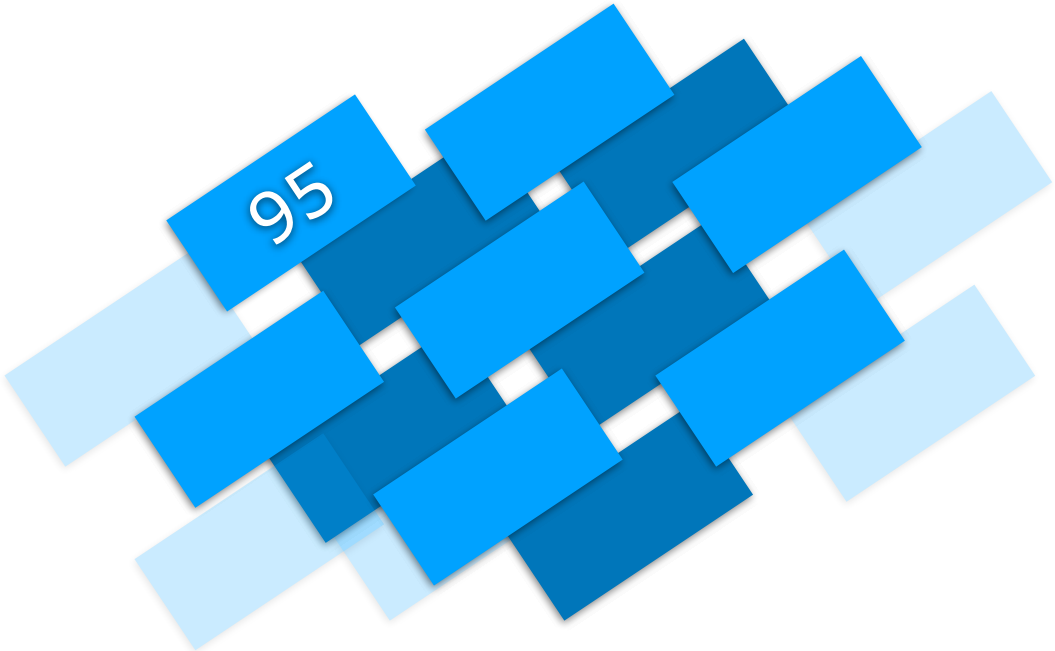
draw is minimum

Imagine I fill a hat with numbered cards and draw one card out at random.

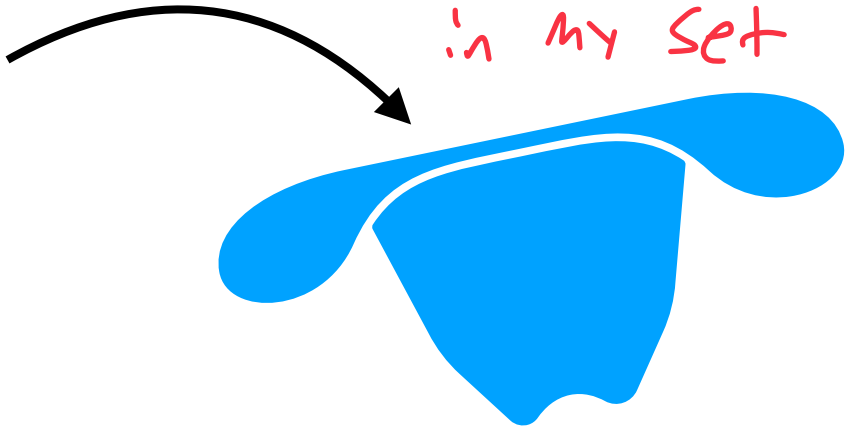
If I told you the value of the card was 95, what have we learned?

↳ 95 is in my set

All other cards ⊆ 95



claim: There are 10 items in my set

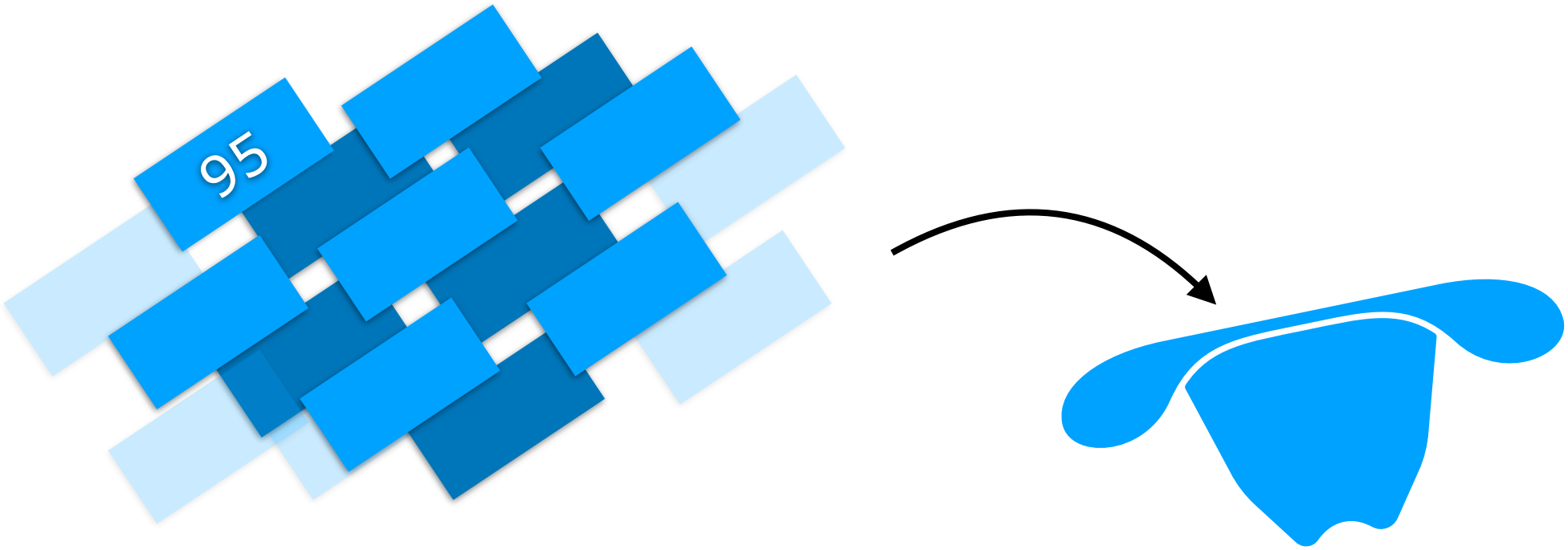


Analogy from Ben Langmead

Cardinality Estimation

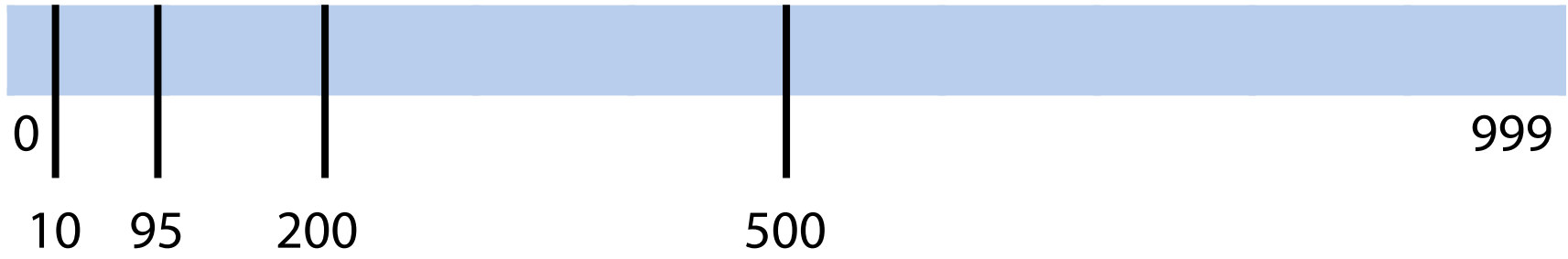
Imagine I fill a hat with **a random subset** of numbered cards **from 0 to 999**

If I told you that the **minimum** value was 95, what have we learned?



Cardinality Estimation

Imagine we have multiple uniform random sets with different minima.



Cardinality Estimation

Let min = 95. Can we estimate N, the cardinality of the set?

Cardinality Estimation

Let min = 95. Can we estimate N, the cardinality of the set?

Claim: $95 \approx \frac{1000}{(N+1)}$

Cardinality Estimation

Let min = 95. Can we estimate N, the cardinality of the set?

Conceptually: If we scatter N points randomly across the interval, we end up with $N + 1$ partitions, each about $1000/(N + 1)$ long

Assuming our first 'partition' is about average:

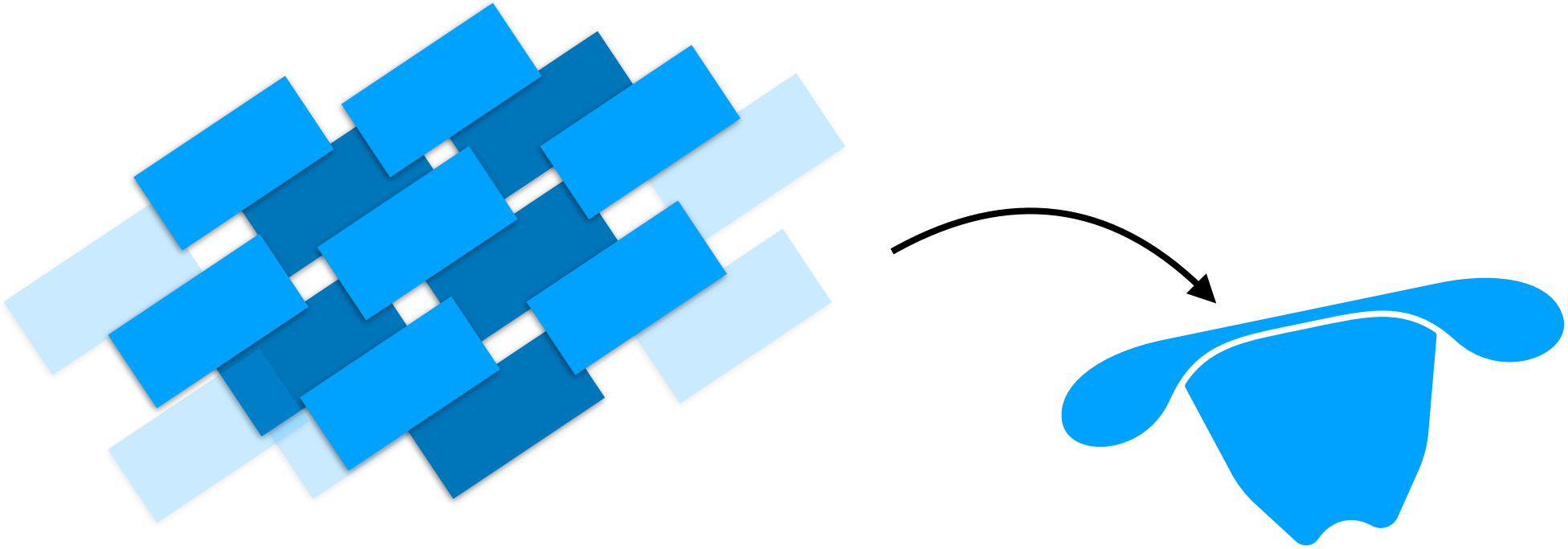
$$95 \approx 1000/(N+1)$$

$$N+1 \approx 10.5$$

$$N \approx 9.5$$

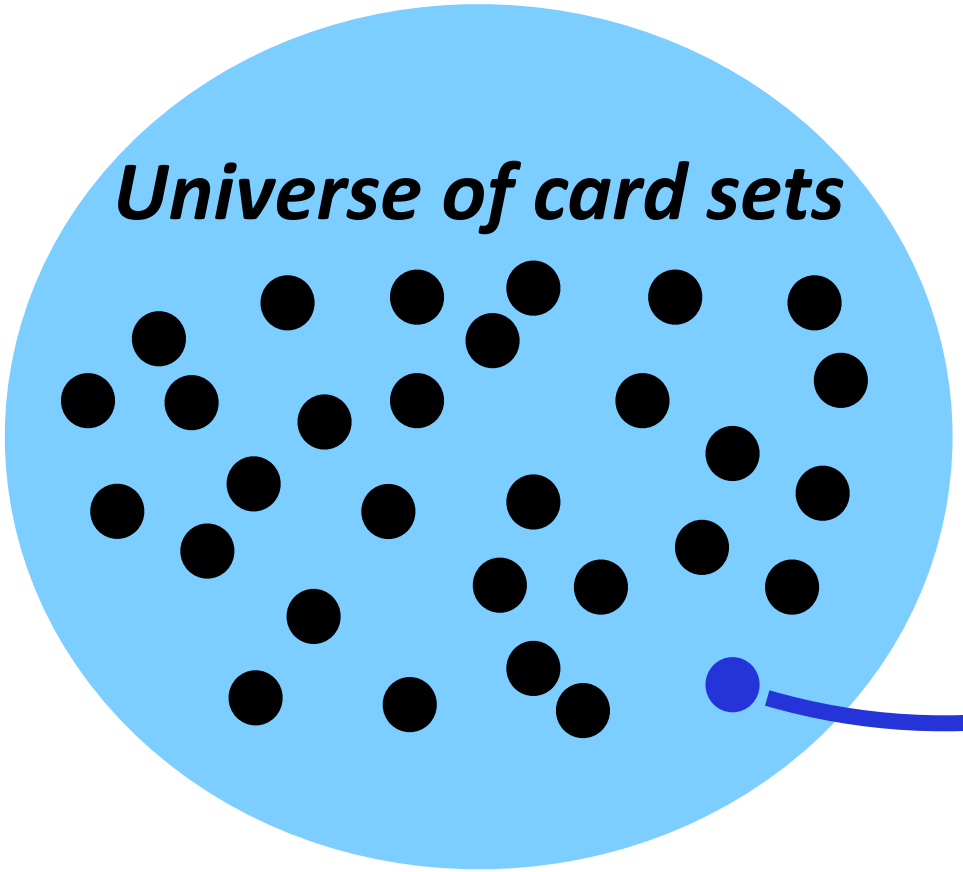
Cardinality Estimation

Why do we care about “the hat problem”?



Cardinality Estimation

Why do we care about “the hat problem”?



m possible minima

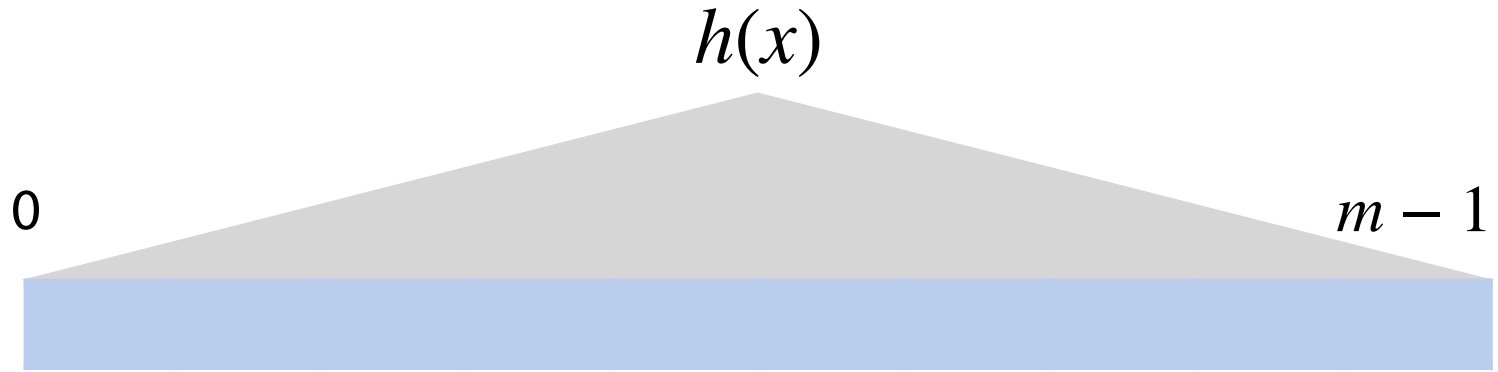
Key	Value

Cardinality Estimation

Imagine we have a SUHA hash h over a range m.

Inserting a new key is equivalent to adding a card to our hat!

Tracking only the minimum value is a **sketch** that estimates the cardinality!



Cardinality Estimation

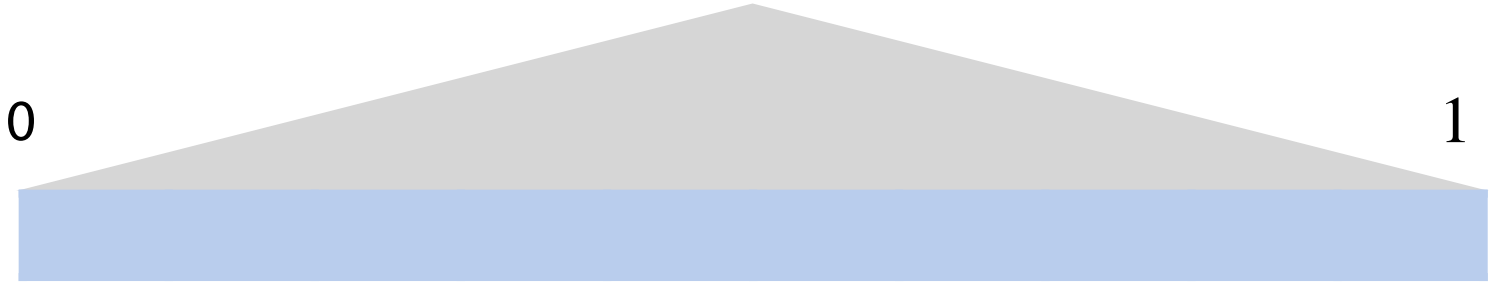
Imagine we have a SUHA hash h over a range m.

Inserting a new key is equivalent to adding a card to our hat!

Tracking only the minimum value is a **sketch** that estimates the cardinality!

To make the math work out, lets normalize our hash…

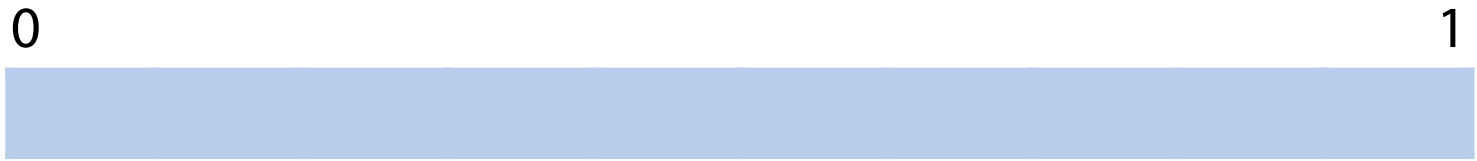
$$h'(x) = h(x) \ / \ (m - 1)$$



Cardinality Sketch

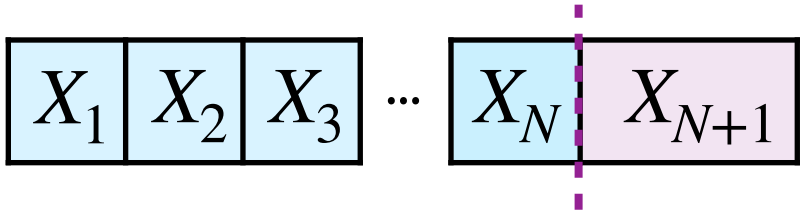
Let $M = min(X_1, X_2, \ldots, X_N)$ where each $X_i \in [0, 1]$ is an uniform independent random variable

Claim: $\mathbf{E}[M] = \frac{1}{N+1}$



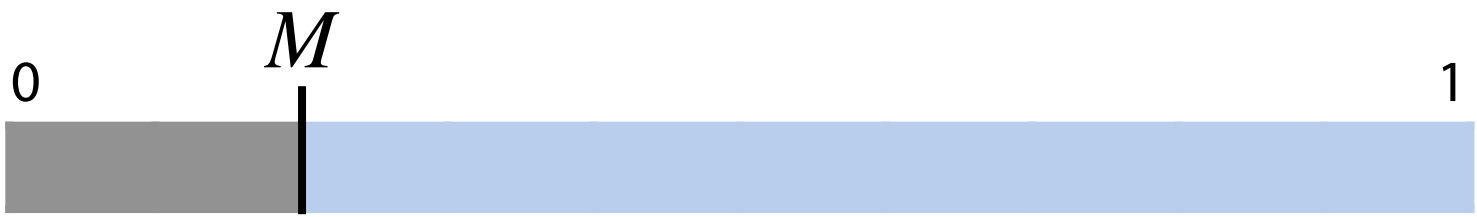
Cardinality Sketch

Consider an $N + 1$ draw:



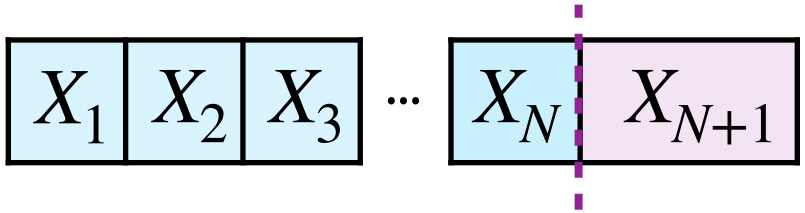
$$M = \min_{1 \le i \le N} X_i$$

X_{N+1} can end up in one of two ranges:



Cardinality Sketch

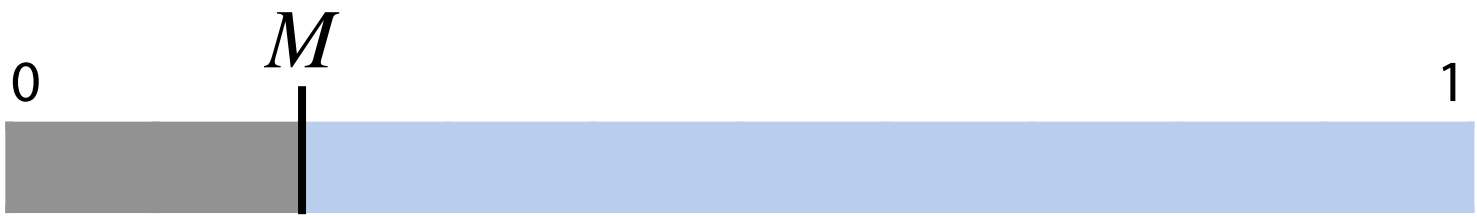
Consider an $N + 1$ draw:



$$M = \min_{1 \le i \le N} X_i$$

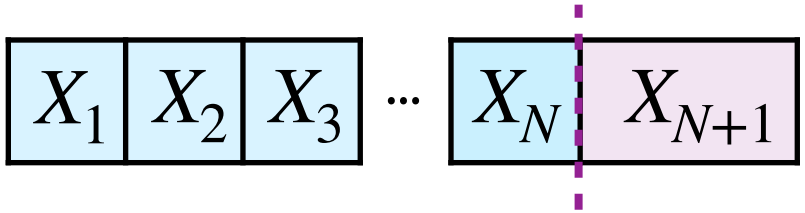
X_{N+1} can end up in one of two ranges:

X_{N+1} will be the new minimum with probability M



Cardinality Sketch

Consider an $N + 1$ draw:

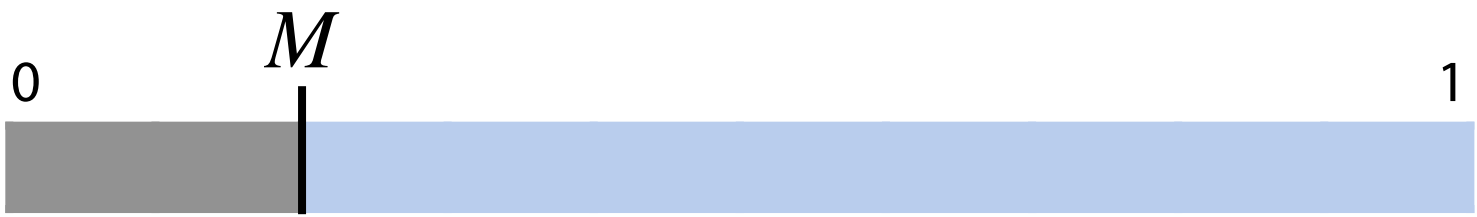


$$M = \min_{1 \le i \le N} X_i$$

X_{N+1} can end up in one of two ranges:

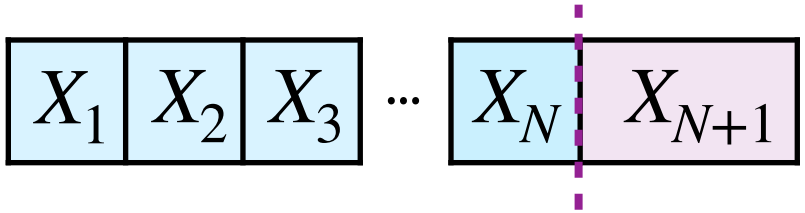
X_{N+1} will be the new minimum with probability M

X_{N+1} will not change minimum with probability $1 - M$



Cardinality Sketch

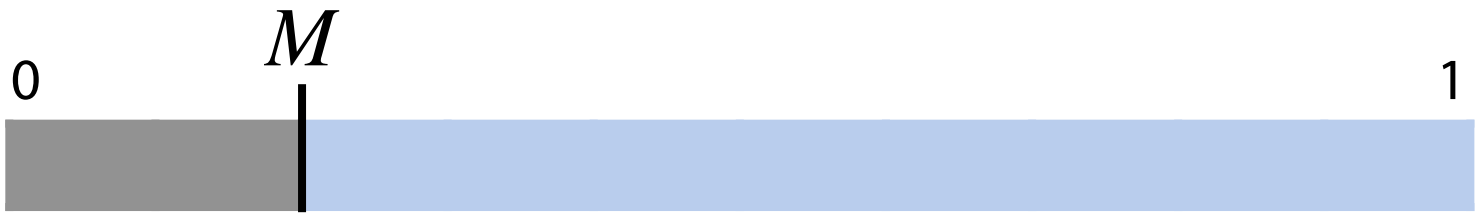
Consider an $N+1$ draw:



$$M = \min_{1 \le i \le N} X_i$$

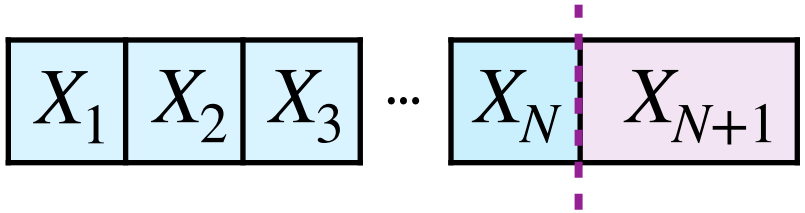
X_{N+1} will be the new minimum with probability M

By definition of SUHA, X_{N+1} has a $\frac{1}{N+1}$ chance of being smallest item



Cardinality Sketch

Consider an $N+1$ draw:

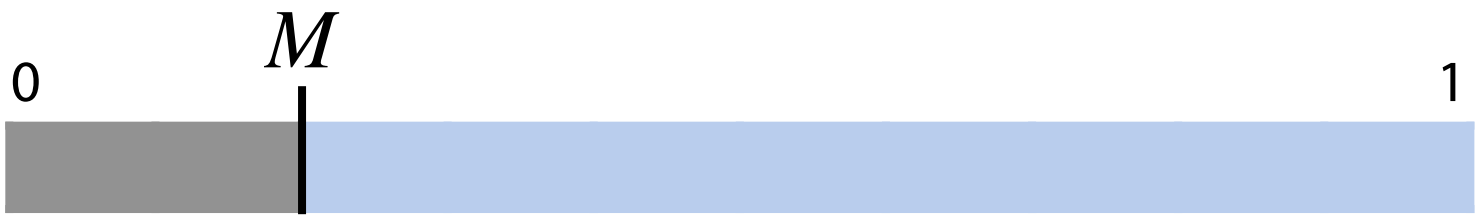


$$M = \min_{1 \le i \le N} X_i$$

X_{N+1} will be the new minimum with probability M

By definition of SUHA, X_{N+1} has a $\frac{1}{N+1}$ chance of being smallest item

Thus, $\mathbf{E}[M] = \frac{1}{N+1}$



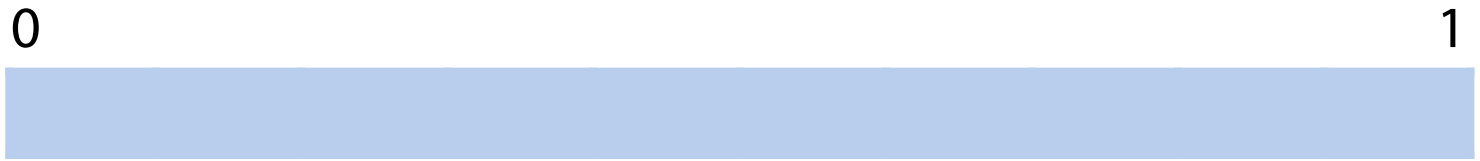
Cardinality Sketch

Claim: $\mathbf{E}[M] = \frac{1}{N+1}$ $\qquad N \approx \frac{1}{M} - 1$

Attempt 1	0.962	0.328	0.771	0.952	0.923
Attempt 2	0.253	0.839	0.327	0.655	0.491
Attempt 3	0.134	0.580	0.364	0.743	0.931

Cardinality Sketch

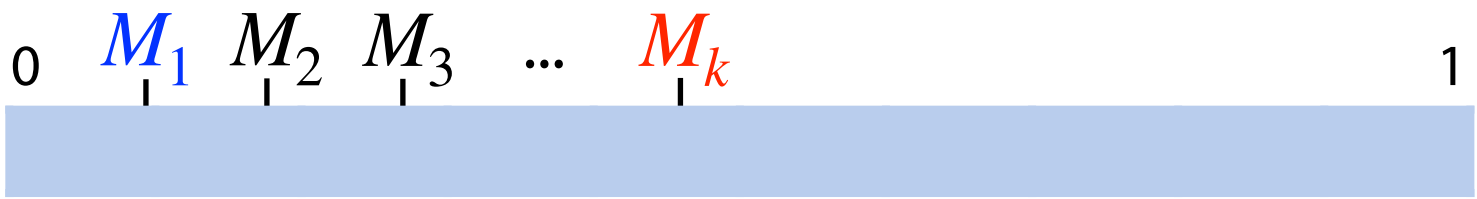
The minimum hash is a valid sketch of a dataset but can we do better?



Cardinality Sketch

Claim: Taking the k^{th}-smallest hash value is a better sketch!

Claim: $\mathbf{E}[M_k] = \dfrac{k}{N+1}$

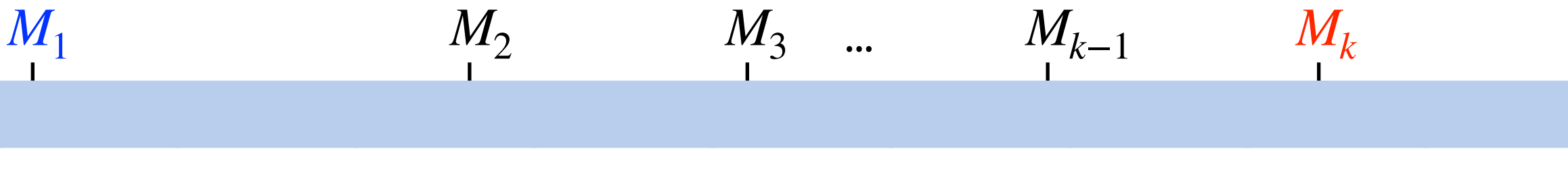


Cardinality Sketch

Claim: Taking the k^{th}-smallest hash value is a better sketch!

Claim: $$\frac{\mathbf{E}[M_k]}{k} = \frac{1}{N+1}$$

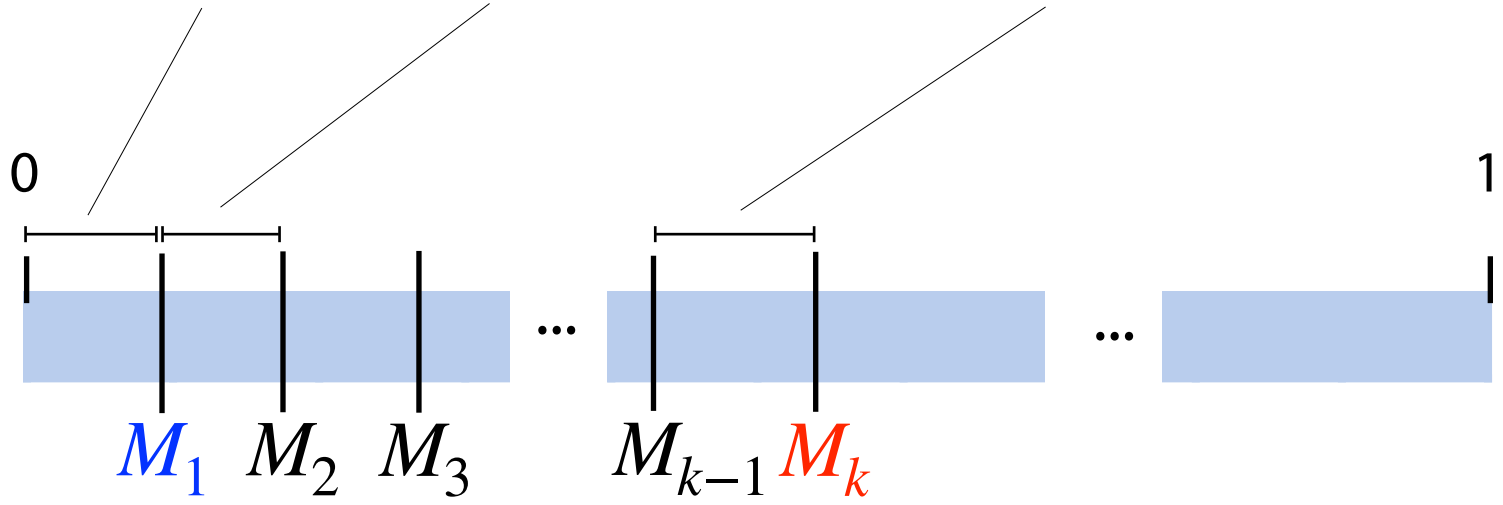
$$= \left[\mathbf{E}[M_1] + (\mathbf{E}[M_2] - \mathbf{E}[M_1]) + \ldots + (\mathbf{E}[M_k] - \mathbf{E}[M_{k-1}])\right] \cdot \frac{1}{k}$$



Cardinality Sketch

$$\frac{1}{N+1} = \frac{\mathbf{E}[M_k]}{k}$$

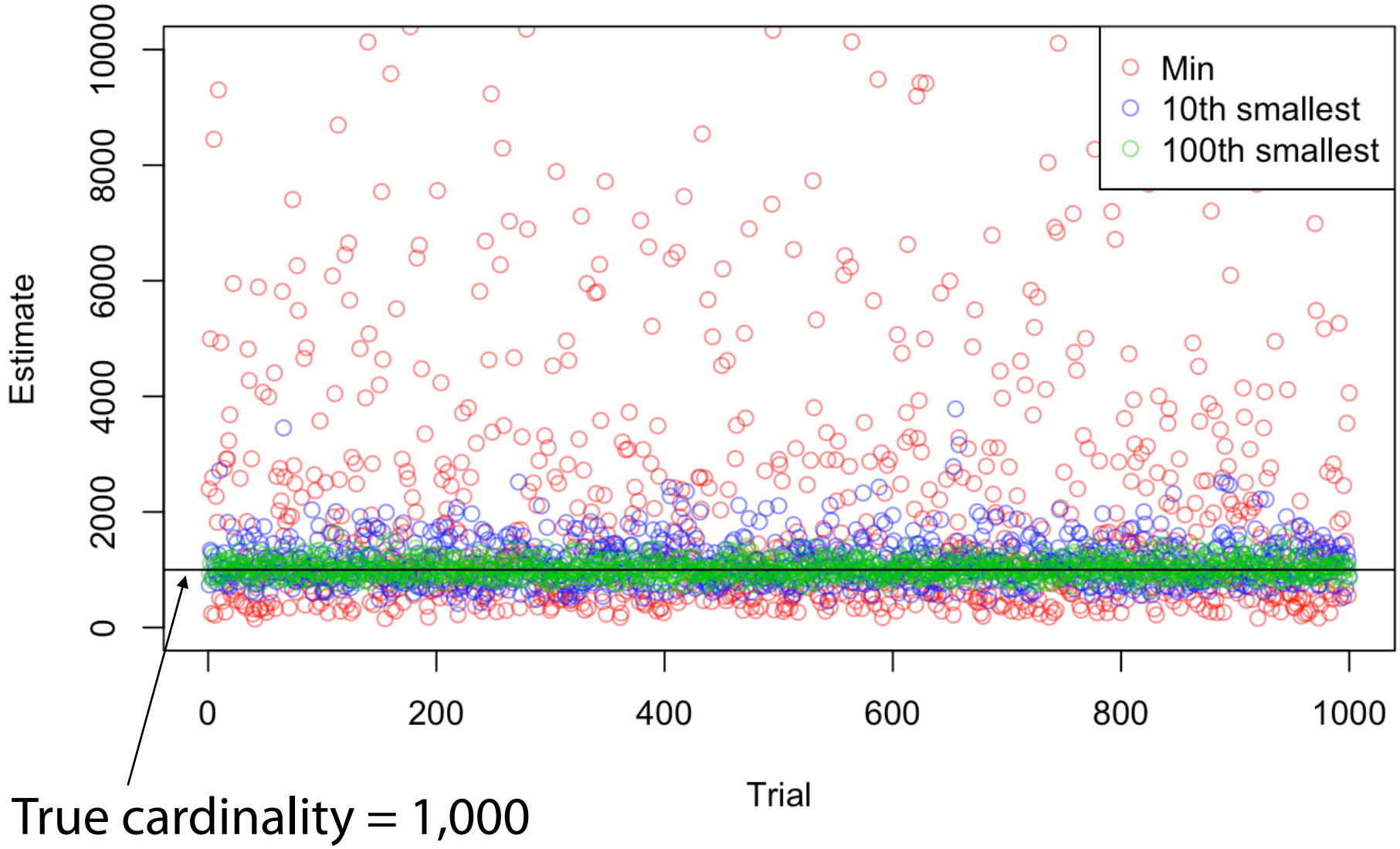
$$= \left[\mathbf{E}[M_1] + (\mathbf{E}[M_2] - \mathbf{E}[M_1]) + \ldots + (\mathbf{E}[M_k] - \mathbf{E}[M_{k-1}])\right] \cdot \frac{1}{k}$$



k^{th} minimum value (KMV)

Averages k estimates for $\frac{1}{N+1}$

Cardinality Sketch



Cardinality Sketch

Given any dataset and a SUHA hash function, we can **estimate the number of unique items** by tracking the **k-th minimum hash value**.

Hash

0.253	0.839	0.327	0.655	0.491

To use the k-th min, we have to track k minima. **Can we use ALL minima?**

Applied Cardinalities

Cardinalities

$|A|$

$|B|$

$|A \cup B|$

$|A \cap B|$

Set similarities

$$O = \frac{|A \cap B|}{min(|A|, |B|)}$$

$$J = \frac{|A \cap B|}{|A \cup B|}$$

Real-world Meaning

```
AGGCCACAGTGTATTATGACTG
|||||||||||  |||||||||
AGGCCACAGTGAGTTATGACTG

AAAAAAAAAAAGATGT-AAGTA
|||||||||||||||| |||||
AAAAAAAAAAAGATGTAAAGTA

GAGG--TCAGATTCACAGCCAC
||||  ||||||||||||||||
GAGGGGTCAGATTCACAGCCAC
```