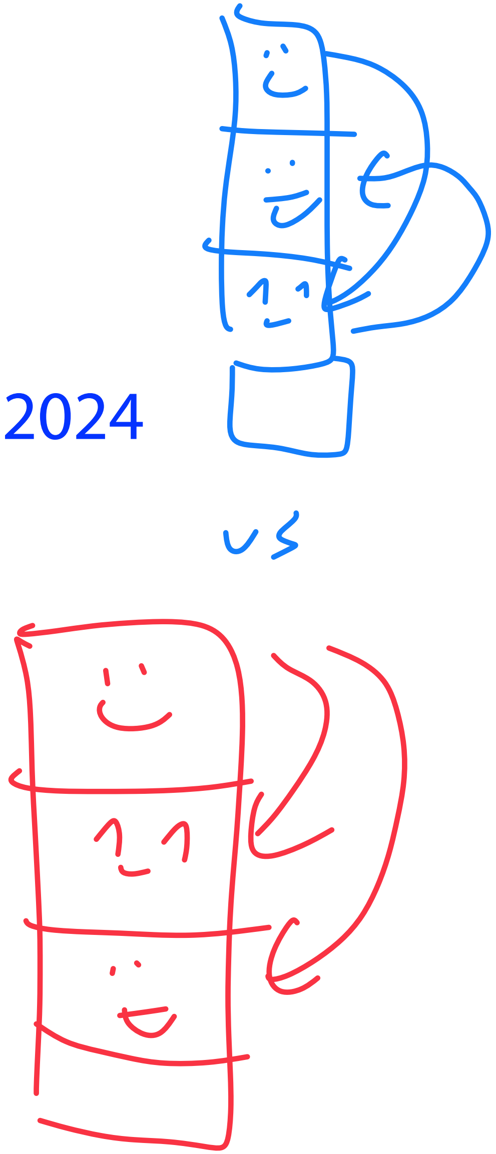
Data Structures and Algorithms

Hashing 3

CS 225
Brad Solomon

November 15, 2024



UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Department of Computer Science

Learning Objectives

Review hash table implementations

Improve our closed hash implementation

Determine when and how to resize a hash table

Justify when to use different index approaches

Simple Uniform Hashing Assumption

Given table of size m, a simple uniform hash, h, implies

$$\forall k_1, k_2 \in U \text{ where } k_1 \neq k_2,\ Pr(h[k_1] = h[k_2]) = \frac{1}{m}$$

Uniform: All keys equally likely to hash to any position

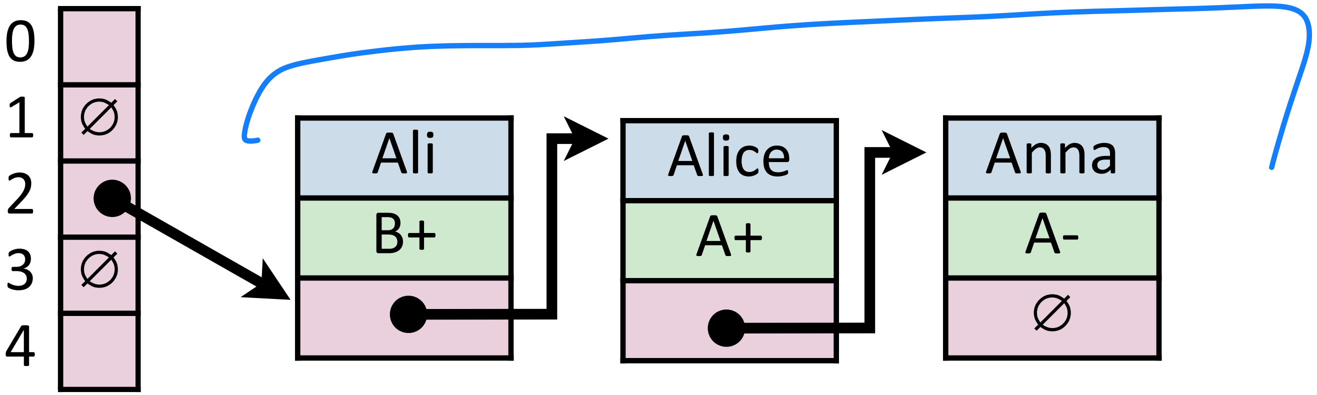
$$Pr(h[k_1]) = \frac{1}{m}$$

Independent: All key's hash values are independent of other keys

Open vs Closed Hashing

Addressing hash collisions depends on your storage structure.

- **Open Hashing:** store *k,v* pairs externally



- **Closed Hashing:** store *k,v* pairs in the hash table

0	Anna, A-
1	
2	Ali, B+
3	Alice, A+

Separate Chaining Under SUHA

Under SUHA, a hash table of size *m* and *n* elements:

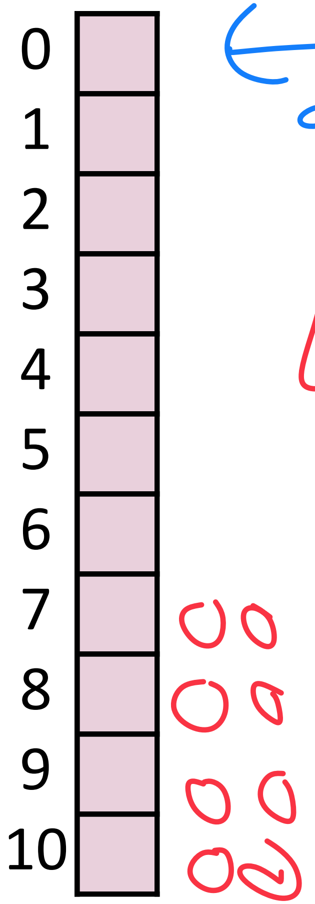
Find runs in: $O(1+\alpha)$.

$\alpha = \frac{n}{m}$

a constant we control

Insert runs in: $O(1)$.

Remove runs in: $O(1+\alpha)$.



expected length

at every position

Load Factor (α)

n/m

(Example of closed hashing)

Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 }

|S| = n

h(k) = k % 7

|Array| = m

0	22
1	8
2	16
3	29
4	4
5	11
6	13

h(k, i) = (k + i) % 7

Try h(k) = (k + $\underline{0}$) % 7, if full…

Try h(k) = (k + $\underline{1}$) % 7, if full…

Try h(k) = (k + $\underline{2}$) % 7, if full…

Try …

Next availabe space $\underline{+1}$

(Example of closed hashing)

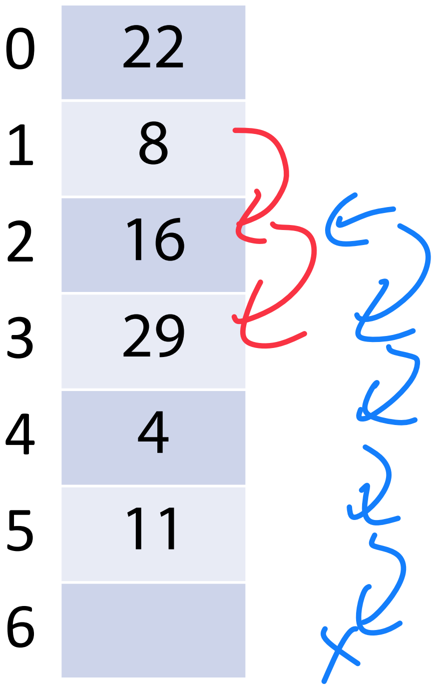
Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 }

|S| = n

h(k, i) = (k + i) % 7

|Array| = m



0	22
1	8
2	16
3	29
4	4
5	11
6	

find(30) 30%7 = 2

`_find(29)`

1) Hash the input key [h(29)=1]

Ideal O(1)

2) Look at hash value (address) position

If present, return (k, v)

If not look at **next available space**

Stop when:

1) We find the object we are looking for

2) We have searched every position in the array

3) We find a blank space

(Example of closed hashing)

Collision Handling: Linear Probing

S = { 16, 8, 4, 13, 29, 11, 22 }

h(k, i) = (k + i) % 7

|S| = n

|Array| = m

0	22	
1	8	
2	~~16~~	1
3	29	
4	4	
5	11	
6	13	

← record that something here once

`_remove(16)`

1) Hash the input key [h(16)=2]
2) Find the actual location (if it exists)
3) Remove the (k,v) at hash value (address)

Don't resize the array! Tombstone!

– find(29)

↳ Blank space stop only if no tombstone

A Problem w/ Linear Probing

Primary Clustering: "Rich get richer"

0	
1	1_1
2	1_2
3	3_1
4	1_3
5	3_2
$\frac{6}{10}$ → 6	
7	
8	
9	

Description:

Collisions create long runs of filled-in indices

Should have a **1/m** chance to hash anywhere

Instead have a **(size of cluster) / m** chance to hash at end

If hash value is 1 → 6

2 → 6

. . .

6 → 6

Remedy:

A Problem w/ Linear Probing

Primary Clustering: “Rich get richer”

0	
1	**1_1**
2	**1_2**
3	**3_1**
4	**1_3**
5	**3_2**
6	
7	
8	
9	

Description:

Collisions create long runs of filled-in indices

Should have a **1/m** chance to hash anywhere

Instead have a **(size of cluster) / m** chance to hash at end

Remedy:

Pick a better “next available” position!

(Example of closed hashing)

Collision Handling: Quadratic Probing

$S = \{ 16, 8, 4, 13, 29, 12, 22 \}$

$h(k) = k \% 7$

$12 \% 7 = 5$

$|S| = n$

$|Array| = m$

One weakness

↳ Can be slower than m to find next available

④	0	12	
	1	**8**	↙ 1
③ →	2	**16**	↙ 1+1
	3	22	
	4	**4**	
$5^{①}$ →	5	29	↙ 1+4
5+1 → ②	6	**13**	

$29 \% 7 = 1$

$(5 + 4) \% 7 = 2$

$(5 + 9) \% 7 = 0$

$h(k, i) = (k + i*i) \% 7$

Try $h(k) = (k + 0) \% 7$, if full…

Try $h(k) = (k + 1*1) \% 7$, if full…

Try $h(k) = (k + 2*2) \% 7$, if full…

Try …

$3 \cdot 3$

$22 \% 7 = 1$

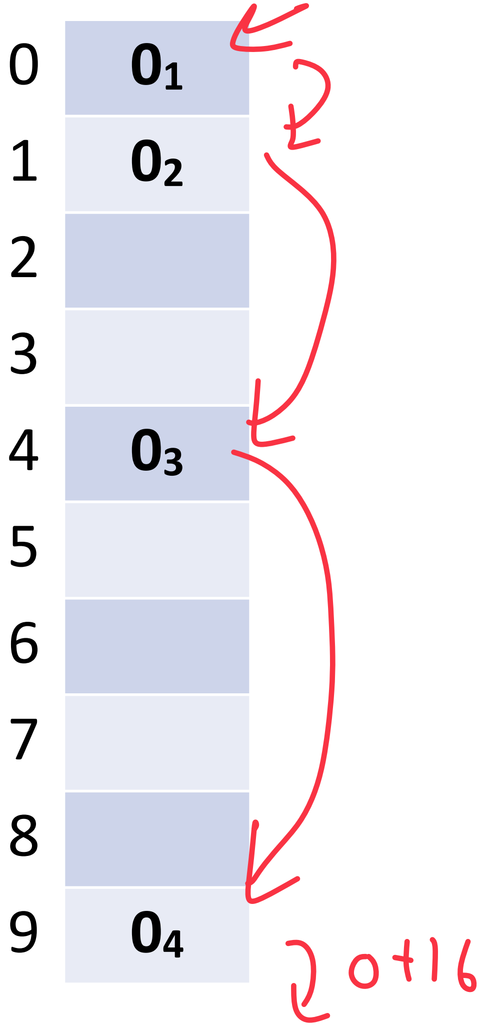
2

$2 + 4$

$1 + 9 = 10 \% 7 = 3$

A Problem w/ Quadratic Probing

Secondary Clustering: We havent solved collisions



Description:

Imagine 4 keys hash to 0

↳ All will systematically collide

(collisions still form long chains over many positions)

Remedy:

↳ Be less consistent but still deterministic

(Example of closed hashing)

Collision Handling: Double Hashing

S = { 16, 8, 4, 13 | 29, 11, 22 }

1 4 1

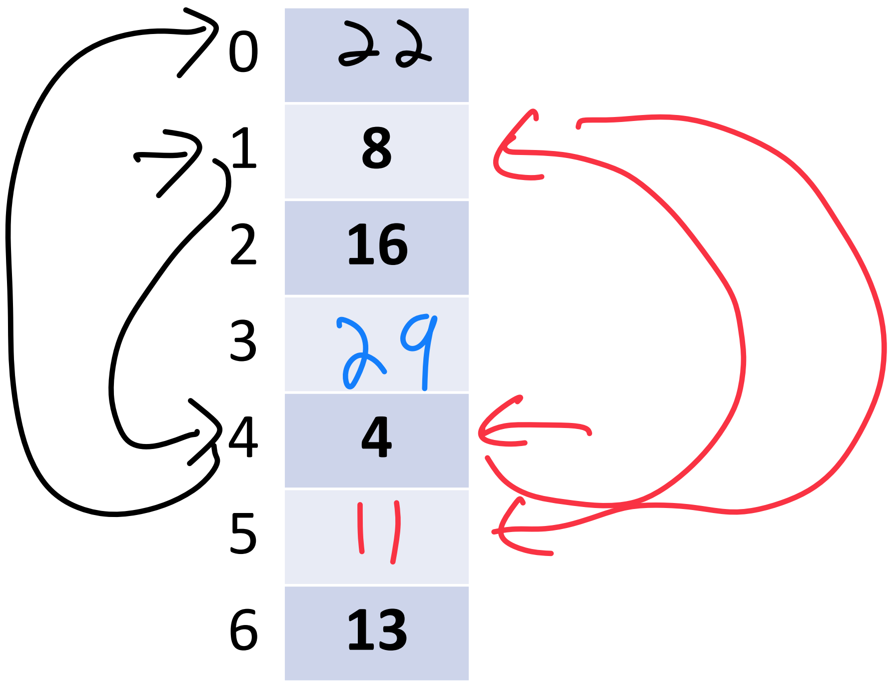
$h_1(k) = k \% 7$

$h_2(k) = 5 - (k \% 5)$ 1 4 3

|S| = n

|Array| = m

To work well
1) M should be prime
2) H_2 needs to be carefu
↳ Be indep of H_1



0	22
1	8
2	16
3	29
4	4
5	11
6	13

$$h(k, i) = (h_1(k) + i*h_2(k)) \% 7$$

Try $h(k) = (k + 0*h_2(k)) \% 7$, if full…

Try $h(k) = (k + 1*h_2(k)) \% 7$, if full…

Try $h(k) = (k + 2*h_2(k)) \% 7$, if full…

Try …

29
1
1+1
1+2

11
4
8%7=1
12%7=5

22
1
(1+3) = 4
1+6 = 7%7=0

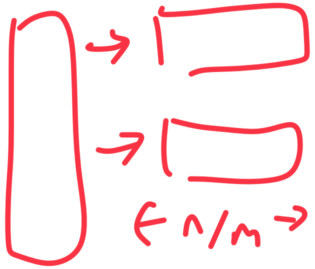
Running Times

(Understand why we have these rough forms)

(Expectation under SUHA)

α = load factor = $\frac{n}{m}$ $\quad [0 \le \alpha \le \infty]$

Open Hashing:



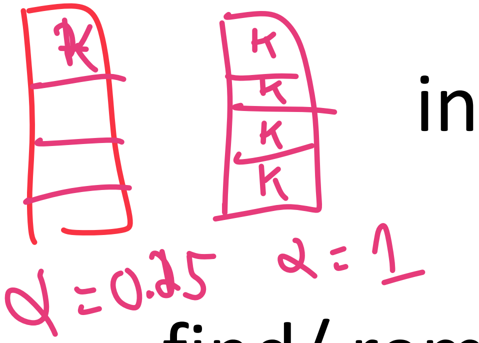
insert: ___ 1 ___.

find/ remove: ___ $1 + \alpha$ ___.

Closed Hashing:

$[0 \le \alpha < 1]$ ← up to m items (n is at most m-1)

α is load factor



insert: ___ $\frac{1}{1-\alpha}$ ___.

find/ remove: ___ $\frac{1}{1-\alpha}$ ___.

Runtime: $\underbrace{1}_{\text{Hash insert}} + \underbrace{1 \cdot \alpha}_{\text{collide once go again}} + \underbrace{1 \cdot \alpha^2}_{\text{collide twice}} + \alpha^3 + \alpha^4 + \dots$

All above $\approx \frac{1}{1-\alpha}$ (taylor series)

Running Times (Expectation under SUHA)

Open Hashing: $0 \leq \alpha \leq \infty$ (Length of chain)

insert: $\underline{\quad 1 \quad}$.

find/ remove: $\underline{\quad 1 + \alpha \quad}$.

Closed Hashing: $0 \leq \alpha < 1$ (fraction full)

insert: $\underline{\quad \frac{1}{1 - \alpha} \quad}$.

find/ remove: $\underline{\quad \frac{1}{1 - \alpha} \quad}$.

Observe:

* **- As α increases:**

OH: $\alpha \to \infty$, runtime $\to \infty$

CH: $\alpha \to 1$, runtime $\to \infty$

* **- If α is constant:**

OH is constant
CH is constant] $O(1)^*$

:)

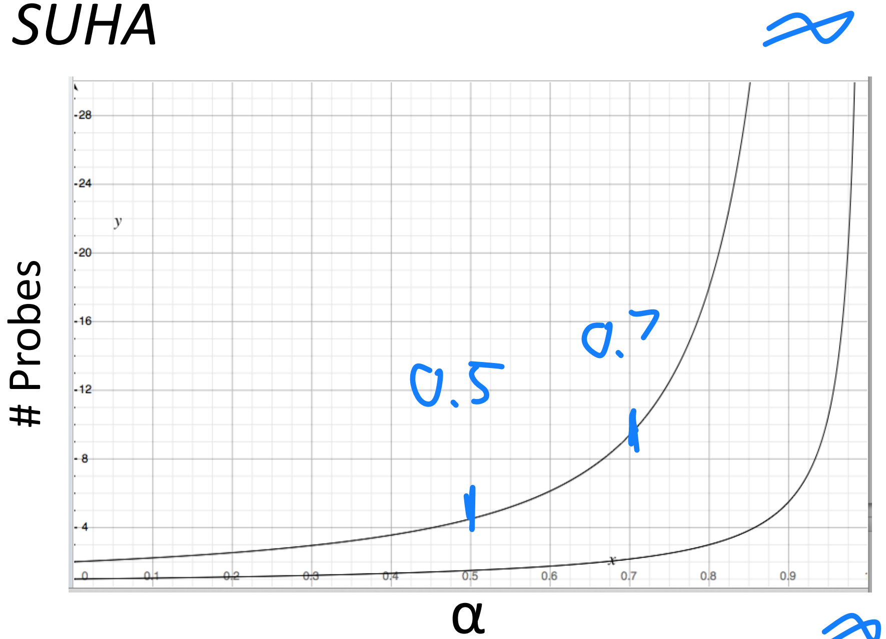
Running Times (Don't memorize these equations, no need.)

The expected number of probes for find(key) under SUHA

Linear Probing:

- Successful: ½(1 + 1/(1-α))
- Unsuccessful: ½(1 + 1/(1-α))²

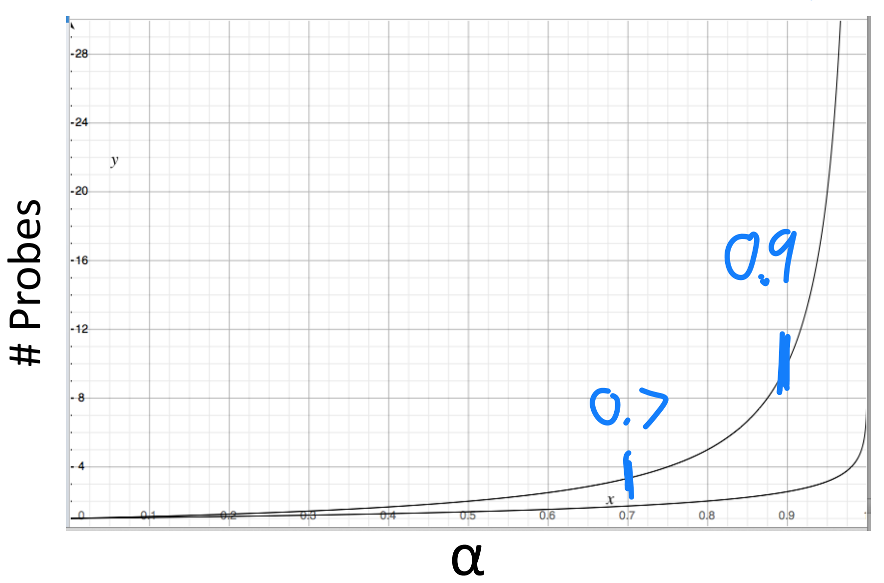
runtime



Double Hashing:

- Successful: 1/α * ln(1/(1-α))
- Unsuccessful: 1/(1-α)

runtime



When do we resize?

Linear ~0,7 - 0,8

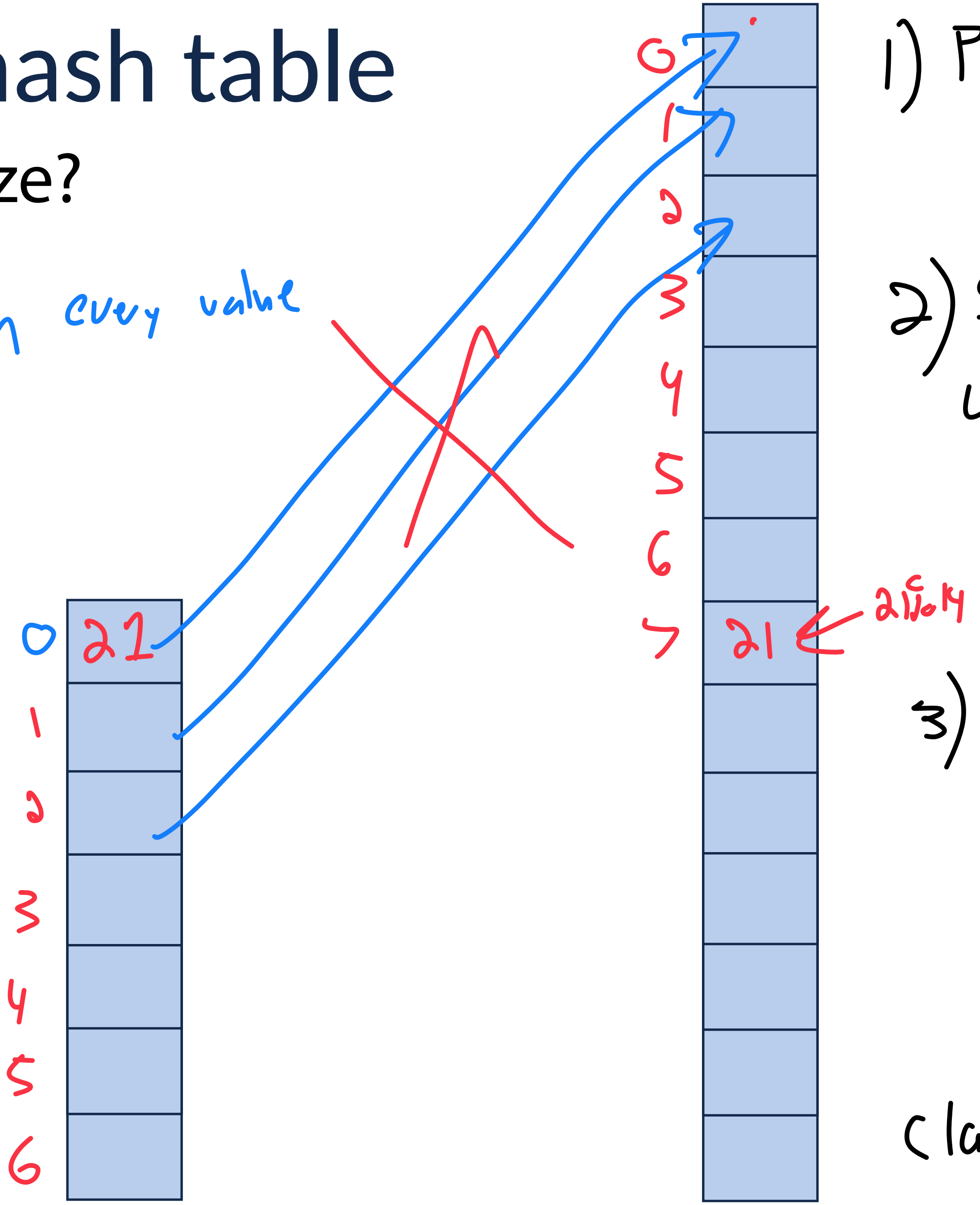
Double ~0.7 - 0,9

Resizing a hash table

How do you resize?

1) Double array size

2) Have to rehash every value



1) Pseudo-amortized

↳ resize every 0.9·n

2) SUHA expectation

↳ Every rehash may be slow

3) Expectation

↳ This is a probability

Claim: O(1)***

Which collision resolution strategy is better?

- Big Records:
- Structure Speed:

What structure do hash tables implement?

What constraint exists on hashing that doesn't exist with BSTs?

Why talk about BSTs at all?

std::map in C++

```
T& map<K, V>::operator[]
pair<iterator, bool> map<K, V>::insert()
iterator map<K, V>::erase()


iterator map<K, V>::lower_bound( const K & );
iterator map<K, V>::upper_bound( const K & );
```

std::unordered_map in C++

```
T& unordered_map<K, V>::operator[]
pair<iterator, bool> unordered_map<K, V>::insert()
iterator unordered_map<K, V>::erase()
```

~~`iterator map<K, V>::lower_bound( const K & );`~~

~~`iterator map<K, V>::upper_bound( const K & );`~~

```
float unordered_map<K, V>::load_factor();
void unordered_map<K, V>::max_load_factor(float m);
```

Running Times

	Hash Table	**AVL**	**Linked List**
Find	Expectation*: $O(1)^{***}$ Worst Case: $O(n)$	$O(\log n)$	$O(n)$
Insert	Expectation*: $O(1)^{***}$ Worst Case: $O(n)$	$O(\log n)$	$O(1)$
Storage Space	$O(n)$	$O(n)$	$O(n)$

Bonus Slides

Hash Table

Worst-Case behavior is bad — but what about randomness?

1) **Fix *h***, our hash, and assume it is good for ***all keys***:

Simple Uniform Hashing Assumption

(Assume our dataset hashes optimally)

2) Create a ***universal hash function family:***

Given a collection of hash functions, pick one randomly

Like **random quicksort** if pick of hash is random, good expectation!

Hash Function (Division Method)

Hash of form: $h(k) = k \,\%\, m$

Pro:

Con:

Hash Function (Mid-Square Method)

Hash of form: $h(k) = (k * k)$ and take b bits from middle $(m = 2^b)$

Hash Function (Mid-Square Method)

Hash of form: $h(k) = (k * k)$ and take b bits from middle $(m = 2^b)$

Hash Function (Multiplication Method)

Hash of form: $h(k) = \lfloor m(kA \,\%\, 1) \rfloor$, $0 \leq A \leq 1$

Pro:

Con:

Hash Function (Universal Hash Family)

Hash of form: $h_{ab}(k) = \big((ak + b)\,\%\,p\big)\,\%\,m,\ a, b \in Z_p^*, Z_p$

$$\forall k_1 \neq k_2,\ Pr_{a,b}(h_{ab}[k_1] = h_{ab}[k_2]) \leq \frac{1}{m}$$

Pro:

Con: