Data Structures and Algorithms
Probability in Computer Science

CS 225
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UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN
Department of Computer Science

Exam 4 (11/13 — 11/15)

Autograded MC and one coding question

Manually graded short answer prompt

Practice exam will be on PL

Topics covered can be found on website

Registration started October 31

https://courses.engr.illinois.edu/cs225/fa2024/exams/

Learning Objectives

Formalize the concept of randomized algorithms

Review fundamentals of probability in computing

Distinguish the three main types of 'random' in computer science

Randomized Algorithms

A **randomized algorithm** is one which uses a source of randomness somewhere in its implementation.

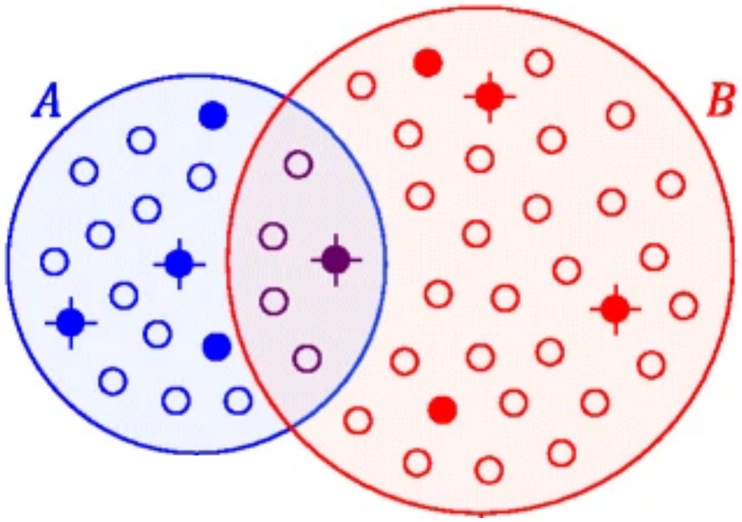
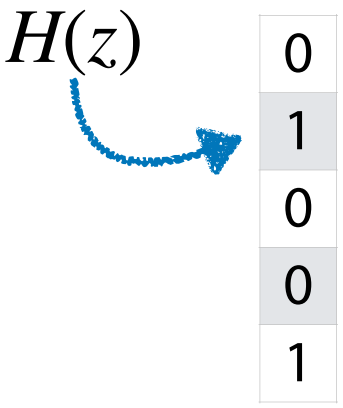
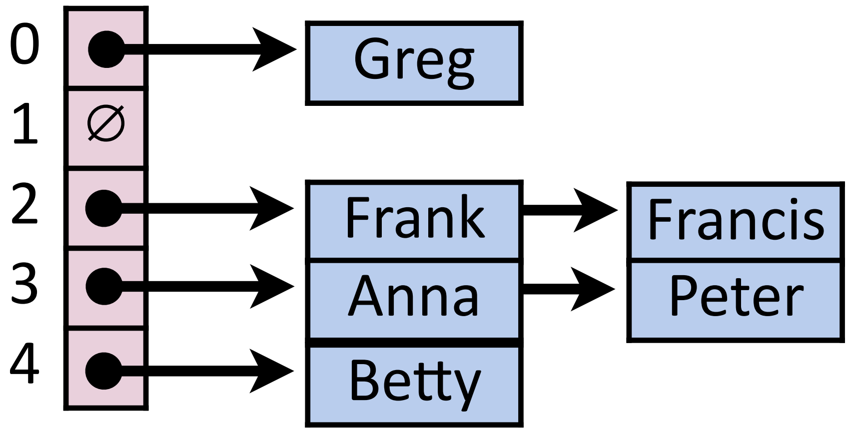


Figure from Ondov et al 2016





$H(x)$	0	2	1	0	0	4	0	2	0	6
$H(y)$	1	0	2	3	1	0	3	4	0	1
$H(z)$	2	1	0	2	0	1	0	0	7	2

A faulty list

Imagine you have a list ADT implementation ***except***…

Every time you called **insert**, it would fail 50% of the time.

Quick Primes with Fermat's Primality Test

If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$

But… ***sometimes*** if n is composite and $a^{n-1} \equiv 1 \pmod{n}$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice.

The **sample space** Ω is the set of all possible outcomes.

An **event** $E \subseteq \Omega$ is any subset.

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

A **random variable** is a function from events to numeric values.

The **expectation** of a (discrete) random variable is:

$$E[X] = \sum_{x \in \Omega} Pr\{X = x\} \cdot x$$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$E[X + Y] = E[X] + E[Y]$ **(Claim)**

Fundamentals of Probability

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$$E[X + Y] = E[X] + E[Y]$$

$$E[X + Y] = \sum_{x} \sum_{y} Pr\{X = x, Y = y\}(x + y)$$

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$$E[X+Y] = E[X] + E[Y]$$

$$E[X+Y] = \sum_{x}\sum_{y} Pr\{X=x, Y=y\}(x+y)$$

$$= \sum_{x} x \sum_{y} Pr\{X=x, Y=y\} + \sum_{y} y \sum_{x} Pr\{X=x, Y=y\}$$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

$$\begin{aligned} E[X+Y] &= \sum_x \sum_y Pr\{X=x, Y=y\}(x+y) \\ &= \sum_x x \sum_y Pr\{X=x, Y=y\} + \sum_y y \sum_x Pr\{X=x, Y=y\} \\ &= \sum_x x \cdot Pr\{X=x\} + \sum_y y \cdot Pr\{Y=y\} \end{aligned}$$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

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Randomization in Algorithms

1. Assume input data is random to estimate average-case performance

2. Use randomness inside algorithm to estimate expected running time

3. Use randomness inside algorithm to approximate solution in fixed time

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Claim: $S(n)$ is $O(n \ log \ n)$

N=3: AllBuild() with every possible permutation of insert order

1 2 3   2 1 3   3 2 1

1 3 2   2 3 1   3 1 2

Average-Case Analysis: BST

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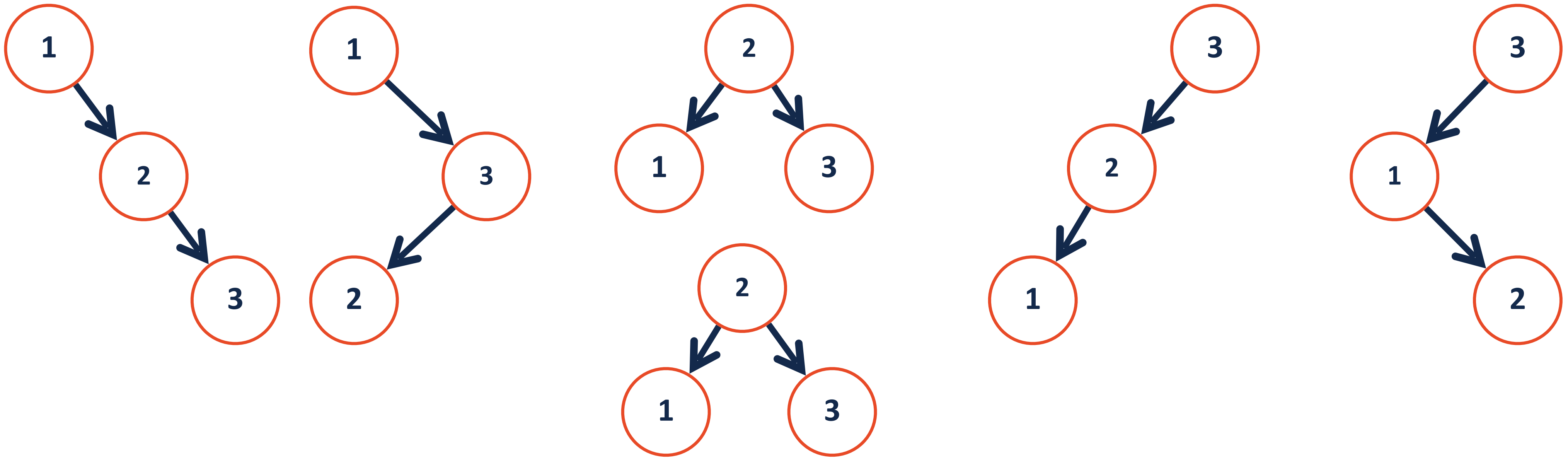
Claim: $S(n)$ is $O(n\ log\ n)$

N=3:



Average-Case Analysis: BST

Let $S(n)$ be the **average** total internal path length **over all BSTs** that can be constructed by uniform random insertion of n objects

Let $0 \le i \le n-1$ be the number of nodes in the left subtree.

Then for a fixed i, $S(n) = (n-1) + S(i) + S(n-i-1)$

$$S(n) = (n-1) + \frac{1}{n}\sum_{i=0}^{n-1} S(i) + S(n-i-1) \approx cn\ ln\ n$$

Here's a slide of math you should not bother learning (in the context of CS 225)

$$S(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} S(i)$$ (1) Guess recurrence form $S(i) = c * i\ ln(i)$

$$S(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} (ci\ ln\ i)$$ (2) Plug in recurrence

$$S(n) \leq (n-1) + \frac{2}{n}\int_{1}^{n} (cx\ ln\ x)dx$$ (3) $\sum_{i=1}^{n-1} f(i) \equiv \int_{1}^{n} f(x)dx$

$$S(n) \leq (n-1) + \frac{2}{n}\left(\frac{cn^2}{2}\ ln\ n - \frac{cn^2}{4} + \frac{c}{4}\right) \approx cn\ ln\ n$$

(4) $\int (cx\ lnx)\ dx$ can be expanded as shown above.

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

$S(n) \approx (n\ \ log\ \ n)$ is provable but a weak argument! **Why?**

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

$S(n) \approx (n \ log \ n)$ is provable but a weak argument! **Why?**

Randomness: Input dataset is considered random

Arguably to extend analysis to 'find' we also assume query is random.

Assumptions: Input dataset is uniform random in content and order

Same assumptions then extended to query

Randomization in Algorithms

1. Assume input data is random to estimate average-case performance

2. Use randomness inside algorithm to estimate expected running time

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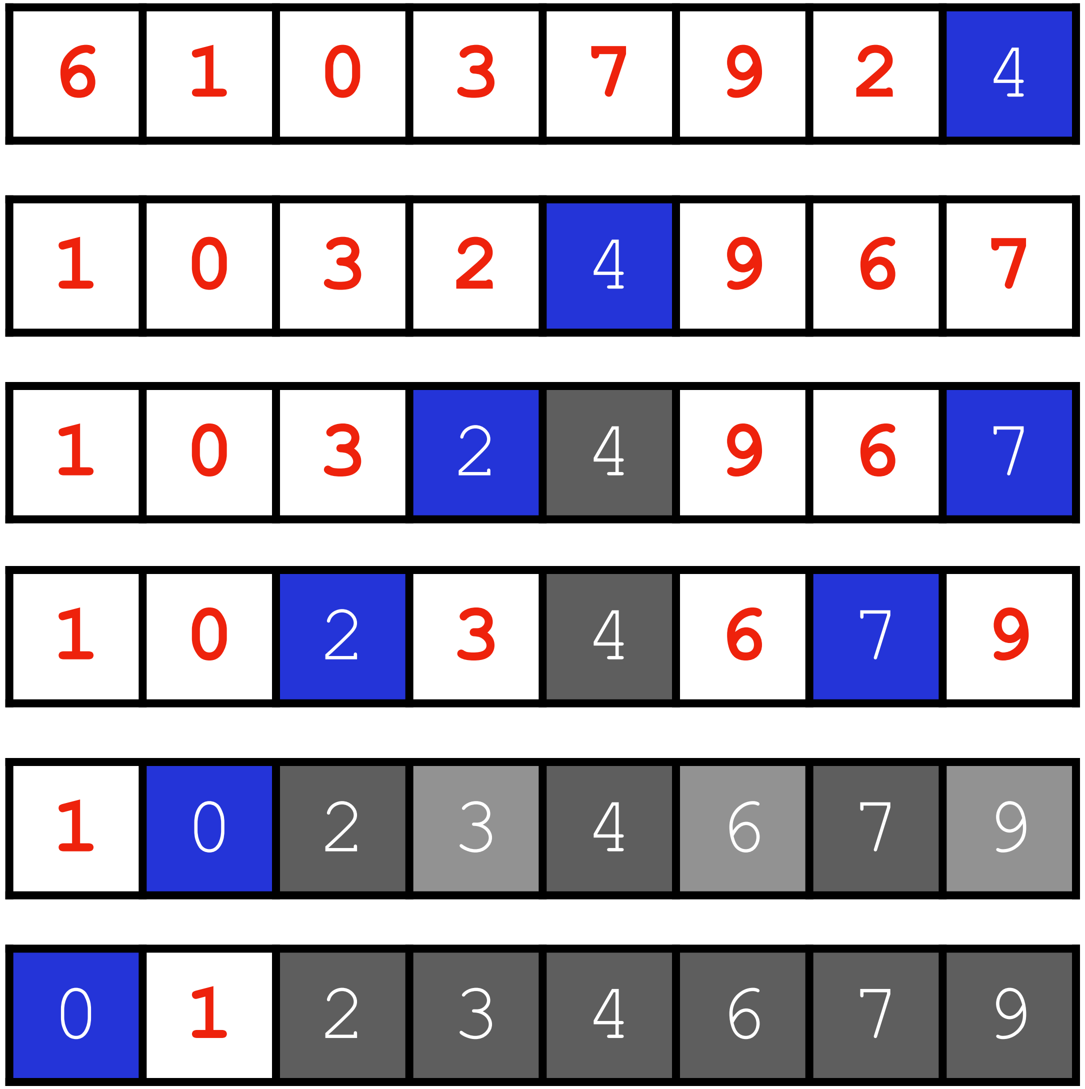
Quicksort Algorithm

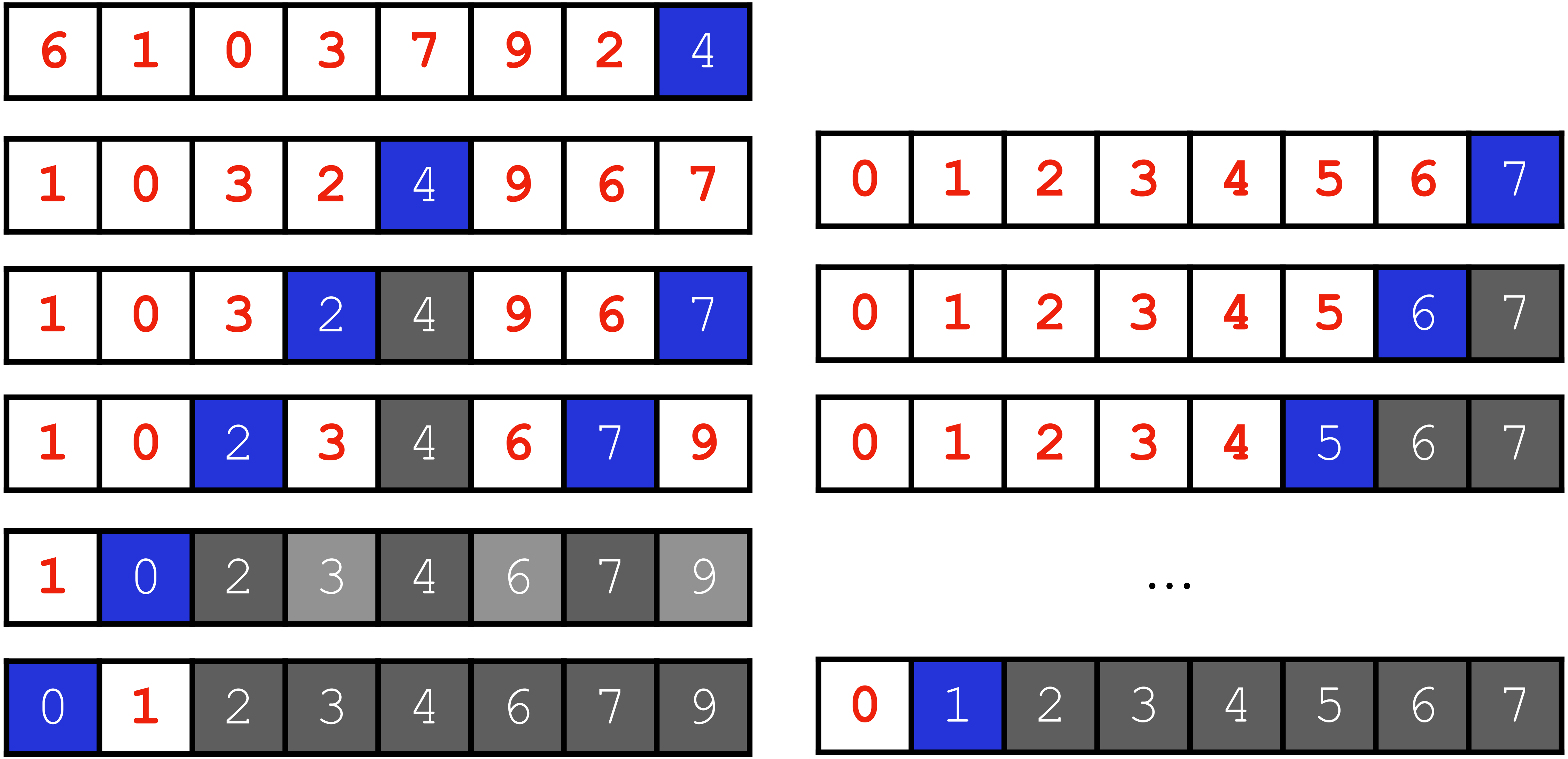


1) Pick Pivot (usually last item)

2) Split array around pivot

3) Recurse on partitions

Problem: Bad pivot leads to bad Big O!



Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected time is $O(n\ log\ n)$ ***for any input!***

Key Idea: We never compare same pair twice!

Proof: Every comparison is against a pivot, but pivot not used in recursion

Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected time is $O(n\ log\ n)$ ***for any input!***

Let X be the total comparisons and X_{ij} be an **indicator variable**:

$$X_{ij} = \begin{cases} 1 & \text{if } i\text{th object compared to } j\text{th} \\ 0 & \text{if } i\text{th object not compared to } j\text{th} \end{cases}$$

Then…

Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

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Then… $$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}$$

We can prove that $E[X] = O(n\ log\ n)$ with a **proof by induction**!

Expectation Analysis: Randomized Quicksort

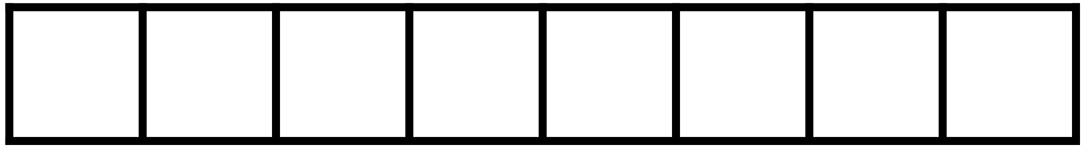
To show $E[X] = O(n\ log\ n)$, we need to first get $E[X_{i,j}]$

Claim: $E[X_{i,j}] = \dfrac{2}{j - i + 1}$.

Base Case: (N=2)

Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \dfrac{2}{j - i + 1}$ **Induction:** Assume true for all inputs of $< n$



Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j - i + 1}$$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j - i + 1}$$

$$E[X] = \sum_{i=1}^{n} 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n - i + 1}\right)$$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n} 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-i+1}\right)$$

$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1) \leq 2n \cdot H_n \leq 2n\ ln\ n$$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j - i + 1}$$

$$E[X] = \sum_{i=1}^{n} 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n - i + 1}\right)$$

(1) Expand out inner sum

$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1)$$

(2) $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$

$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1) \leq 2n \cdot H_n \leq 2n\ ln\ n$$

(3) $H_n = \theta(log\ n)$

Expectation Analysis: Randomized Quicksort

Summary: Randomized quick sort is $O(n \ log \ n)$ regardless of input

Randomness:

Assumptions:

Randomization in Algorithms

1. Assume input data is random to estimate average-case performance

2. Use randomness inside algorithm to estimate expected running time

3. Use randomness inside algorithm to approximate solution in fixed time

Probabilistic Accuracy: Fermat primality test

Pick a random a in the range $[2,\ p-2]$

If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$

But… ***sometimes*** if n is composite and $a^{n-1} \equiv 1 \pmod{n}$

Probabilistic Accuracy: Fermat primality test

	$a^{p-1} \equiv 1 \pmod{p}$	$a^{p-1} \not\equiv 1 \pmod{p}$
p is prime		
p is not prime		

Probabilistic Accuracy: Fermat primality test

Let's assume $\alpha = .5$

First trial: $a = a_0$ and prime test returns 'prime!'

Second trial: $a = a_1$ and prime test returns 'prime!'

Third trial: $a = a_2$ and prime test returns 'not prime!'

Is our number prime?

What is our **false positive** probability? Our **false negative** probability?

Probabilistic Accuracy: Fermat primality test

Summary: Randomized algorithms can also have fixed (or bounded) runtimes at the cost of probabilistic accuracy.

Randomness:

Assumptions:

Types of randomized algorithms

A **Las Vegas** algorithm is a randomized algorithm which will always give correct answer if run enough times but has no fixed runtime.

A **Monte Carlo** algorithm is a randomized algorithm which will run a fixed number of iterations and may give the correct answer.

Next Class: Randomized Data Structures

Sometimes a data structure can be **too ordered / too structured**

Randomized data structures rely on **expected** performance

Randomized data structures 'cheat' tradeoffs!