Data Structures and Algorithms
Probability in Computer Science

CS 225
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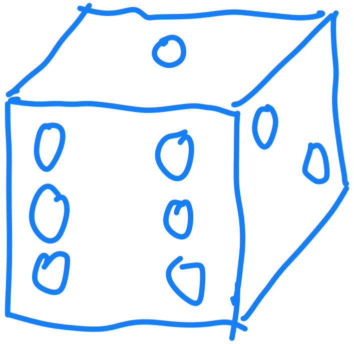
November 8, 2024



Department of Computer Science

Exam 4 (11/13 — 11/15)

Autograded MC and one coding question

Manually graded short answer prompt

Practice exam will be on PL

Topics covered can be found on website

Registration started October 31

https://courses.engr.illinois.edu/cs225/fa2024/exams/

Learning Objectives

Formalize the concept of randomized algorithms

Review fundamentals of probability in computing

Distinguish the three main types of 'random' in computer science

Randomized Algorithms

A **randomized algorithm** is one which uses a source of randomness somewhere in its implementation.

Bloom Filter

Hash Table

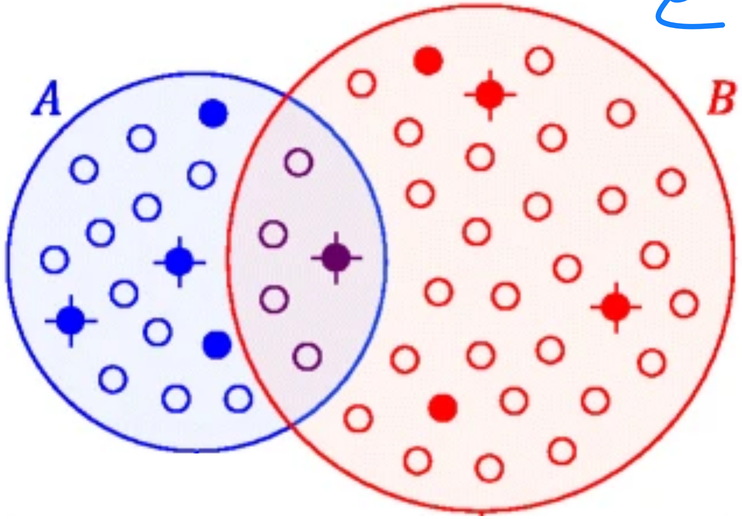
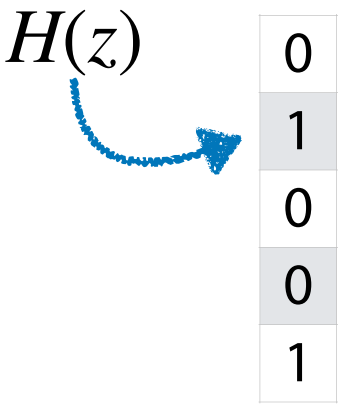
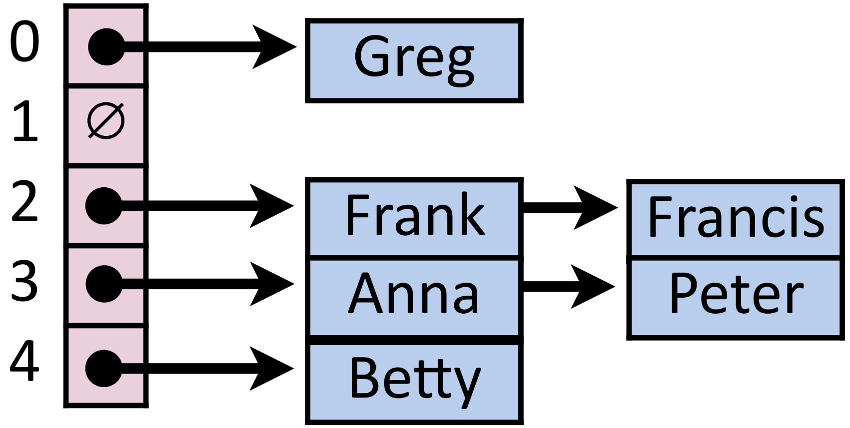


Figure from Ondov et al 2016

MinHash Sketch





$H(x)$	0	2	1	0	0	4	0	2	0	6
$H(y)$	1	0	2	3	1	0	3	4	0	1
$H(z)$	2	1	0	2	0	1	0	0	7	2

Skip List

A faulty list

Imagine you have a list ADT implementation ***except***…

Every time you called **insert**, it would fail 50% of the time.

↳ Could use to simulate repeated coin flips

↳ webpage cache

↳ Counting in an ∞ data stream

↳ Probabilistic data cleaning

Real items have high repetition and fake/errors dont

Quick Primes with Fermat's Primality Test

$[2, p-2]$

If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$

But… ***sometimes*** if n is composite and $a^{n-1} \equiv 1 \pmod{n}$

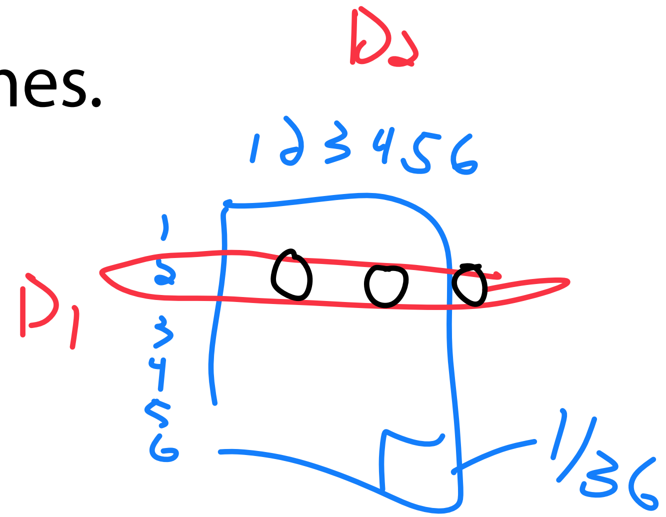
If $a=2$, $\frac{21,853}{25 \cdot 10^9 \text{ integers}}$ "pseudoprimes" – not prime but pass test

Not 100% accurate but… 99.9% accurate & fast

Fundamentals of Probability

Imagine you roll a pair of six-sided dice.

The **sample space** Ω is the set of all possible outcomes.



An **event** $E \subseteq \Omega$ is any subset.

D1 rolls 2 and D2 is even

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

A **random variable** is a function from events to numeric values.

D1 is value of first dice →

DBoth is value of D1 + D2

The **expectation** of a (discrete) random variable is:

$$E[D1] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \dots \approx 3.5$$

$$E[X] = \sum_{x \in \Omega} Pr\{X = x\} \cdot x$$

$$E[DBoth] = \frac{1}{36} \cdot 2 + \frac{1}{36} \cdot (1+2) + \dots \approx 7$$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$E[X + Y] = E[X] + E[Y]$ **(Claim)**

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$$E[X + Y] = E[X] + E[Y]$$

$$E[X + Y] = \sum_x \sum_y \underbrace{Pr\{X = x, Y = y\}}_{\text{prob of event}}(x + y)$$

∘ value of event

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

Sum of probabilities is 1

$$E[X+Y] = \sum_x \sum_y Pr\{X=x, Y=y\}(x+y)$$

$$= \sum_x x \cancel{\sum_y} Pr\{X=x, Y \cancel{= y}\} + \sum_y y \cancel{\sum_x} Pr\{X \cancel{= x}, Y=y\}$$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

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$$E[X+Y] = E[X] + E[Y]$$

$$E[X+Y] = \sum_x \sum_y Pr\{X=x, Y=y\}(x+y)$$

$$= \sum_x x \sum_y Pr\{X=x, Y=y\} + \sum_y y \sum_x Pr\{X=x, Y=y\}$$

$$= \sum_x x \cdot Pr\{X=x\} + \sum_y y \cdot Pr\{Y=y\}$$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

Linearity of Expectation: For any two random variables X and Y,

$$E[X + Y] = \underbrace{E[X]}_{3.5} + \underbrace{E[Y]}_{3.5}$$

$= 7$

Randomization in Algorithms

1. Assume input data is random to estimate average-case performance

2. Use randomness inside algorithm to estimate expected running time

3. Use randomness inside algorithm to approximate solution in fixed time

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Claim: $S(n)$ is $O(n \ log \ n)$

N=3: AllBuild() with every possible permutation of insert order

1 2 3   2 1 3   3 2 1

1 3 2   2 3 1   3 1 2

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

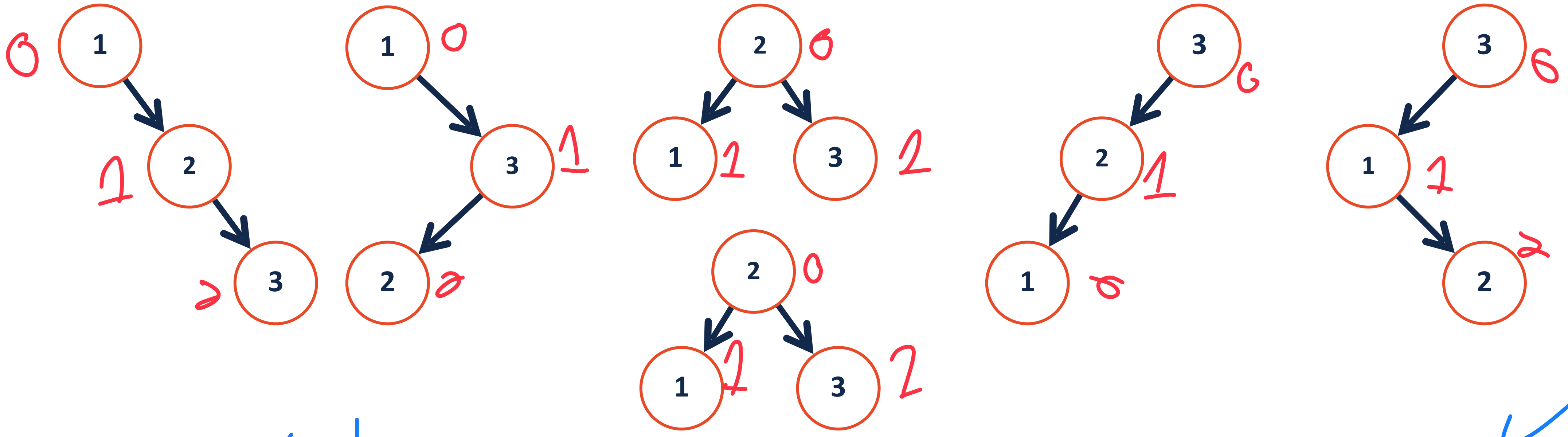
Claim: $S(n)$ is $O(n\ log\ n)$

path length

$6 \cdot 0 + 8 \cdot 1 + 4 \cdot 2 = 16/6 \approx 2.66$

trees

N=3:



$3 \log 3 \sim 4.75$

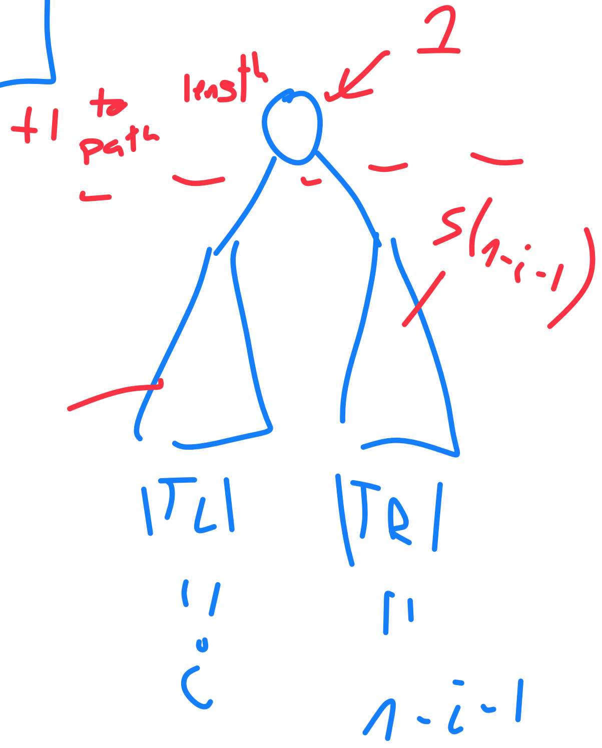
Average-Case Analysis: BST

Let $S(n)$ be the **average** total internal path length **over all BSTs** that can be constructed by uniform random insertion of n objects

Let $0 \le i \le n-1$ be the number of nodes in the left subtree.

Then for a fixed i, $S(n) = (n-1) + S(i) + S(n-i-1)$

$$S(n) = (n-1) + \frac{1}{n}\sum_{i=0}^{n-1} S(i) + S(n-i-1) \approx cn\ ln\ n$$



S(i)

n-1 items here

Here's a slide of math you should not bother learning (in the context of CS 225)

$$S(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} S(i)$$

(1) Guess recurrence form $S(i) = c * i \ ln(i)$

$$S(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} (ci \ ln \ i)$$

(2) Plug in recurrence

$$S(n) \leq (n-1) + \frac{2}{n}\int_{1}^{n} (cx \ ln \ x)dx$$

(3) $\sum_{i=1}^{n-1} f(i) \equiv \int_{1}^{n} f(x)dx$

$$S(n) \leq (n-1) + \frac{2}{n}\left(\frac{cn^2}{2} \ ln \ n - \frac{cn^2}{4} + \frac{c}{4}\right) \approx cn \ ln \ n$$

(4) $\int (cx \ lnx) \ dx$ can be expanded as shown above.

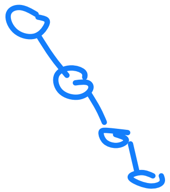
Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

$S(n) \approx (n \; log \; n)$ is provable but a weak argument! **Why?**

Average-Case Analysis: BST

real $O(n)$



Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

$S(n) \approx (n \; log \; n)$ is provable but a weak argument! **Why?**

Randomness: Input dataset is considered random

Arguably to extend analysis to 'find' we also assume query is random.

Assumptions: Input dataset is uniform random in content and order

Same assumptions then extended to query

Randomization in Algorithms

1. Assume input data is random to estimate average-case performance

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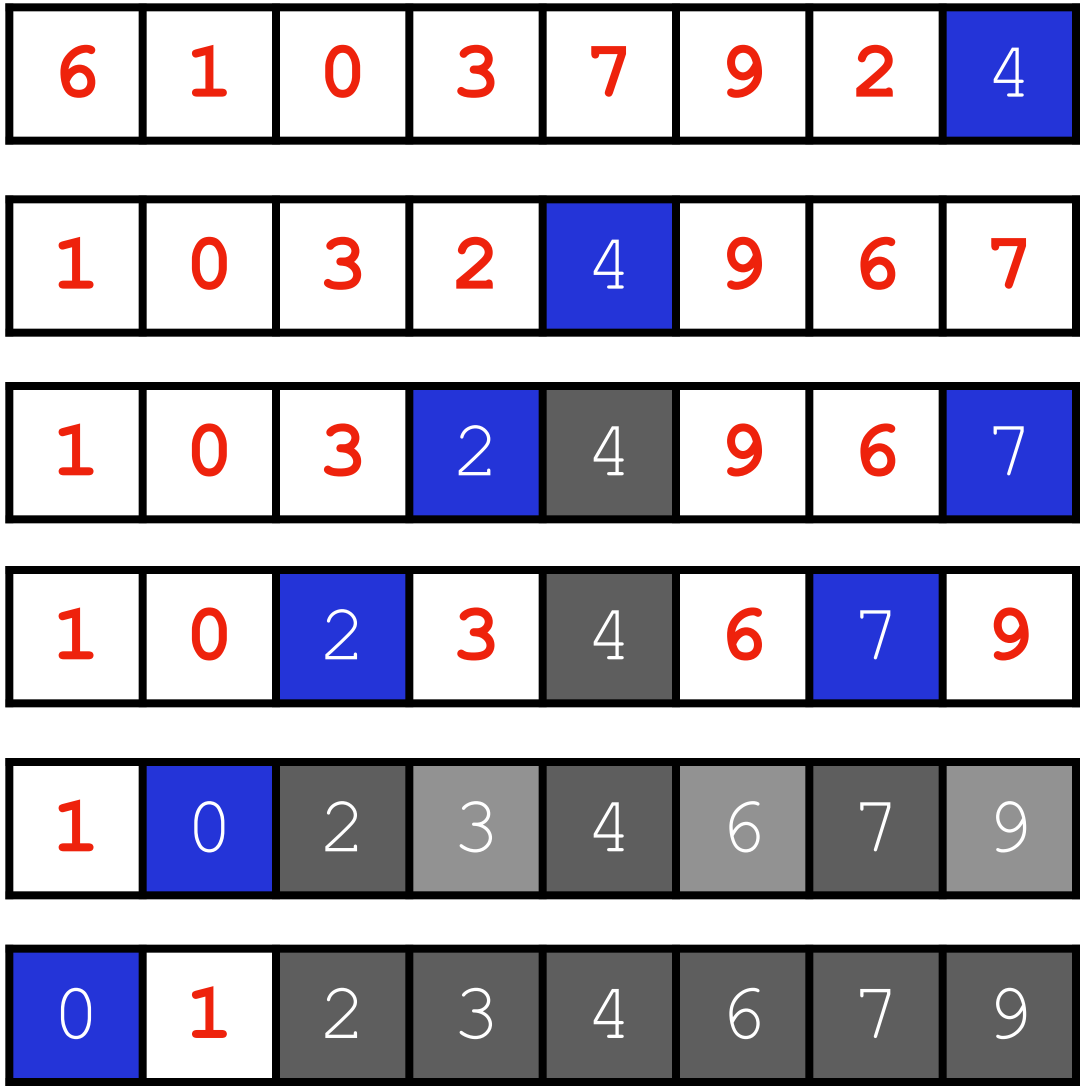
Quicksort Algorithm

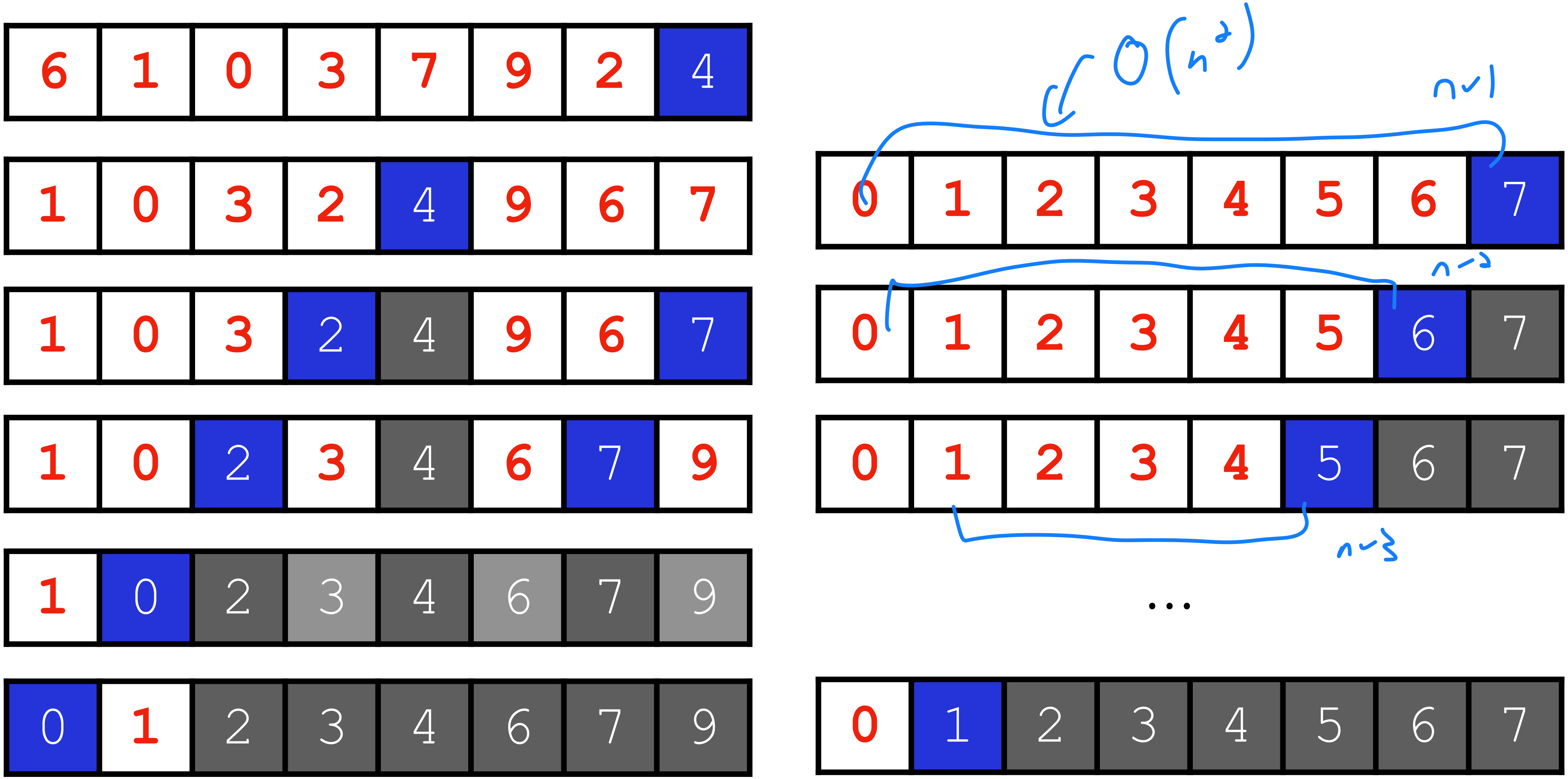


1) Pick Pivot (usually last item)

2) Split array around pivot

3) Recurse on partitions

Problem: Bad pivot leads to bad Big O!



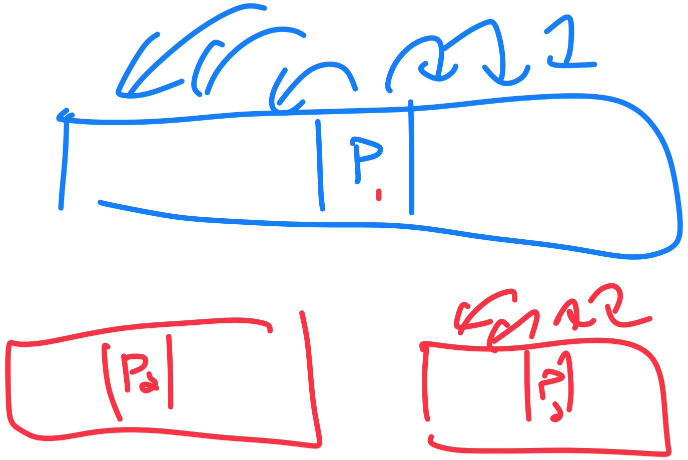
Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected time is $O(n\ log\ n)$ ***for any input!***

Key Idea: We never compare same pair twice!

Proof: Every comparison is against a pivot, but pivot not used in recursion



Pivot compares all items

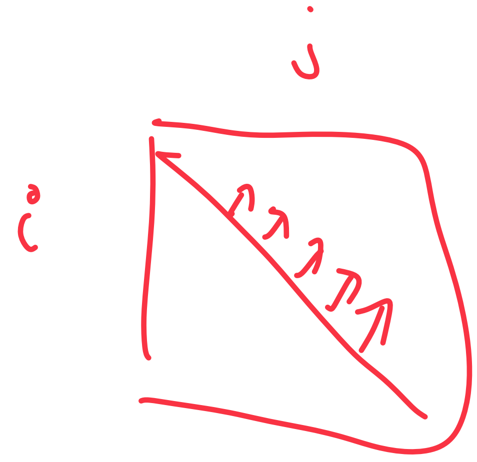
Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected time is $O(n\ \ log\ \ n)$ ***for any input!***

Let X be the total comparisons and X_{ij} be an **indicator variable**:

$$X_{ij} = \begin{cases} 1 & \text{if } i\text{th object compared to } j\text{th} \\ 0 & \text{if } i\text{th object not compared to } j\text{th} \end{cases}$$



Then… $X = \sum_{i} \sum_{j} x_{ij}$

$i = [1, n]$

index start @ 1

$j = [i+1, n]$

Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected time is $O(n \ log \ n)$ ***for any input!***

Let X be the total comparisons and X_{ij} be an **indicator variable**:

$$X_{ij} = \begin{cases} 1 & \text{if } i\text{th object compared to } j\text{th} \\ 0 & \text{if } i\text{th object not compared to } j\text{th} \end{cases}$$

Then… $$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}$$

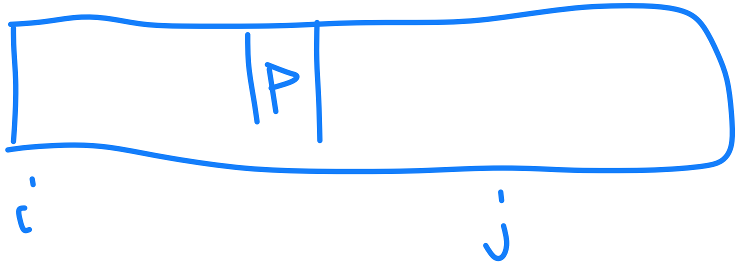
$\doteq E[X_{ij}]$

We can prove that $E[X] = O(n \ log \ n)$ with a **proof by induction**!

Expectation Analysis: Randomized Quicksort

To show $E[X] = O(n\ log\ n)$, we need to first get $E[X_{i,j}]$

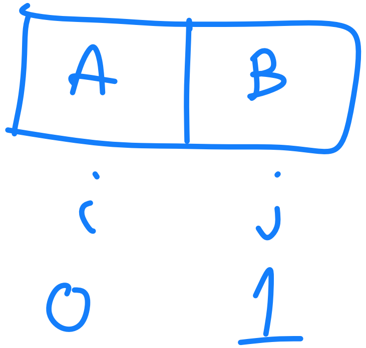
Claim: $E[X_{i,j}] = \dfrac{2}{j-i+1}$.



Base Case: (N=2) b/c j = i+1 as min index

Eq

$$\frac{2}{1-0+1} = \frac{2}{2} = 1$$



Real

1 comparison

P=A

1

B

P=B

A

Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \dfrac{2}{j-i+1}$ **Induction:** Assume true for all inputs of $< n$

$= \Pr[X_{i,j}=1 \mid j<p] \cdot \Pr[j<p]$

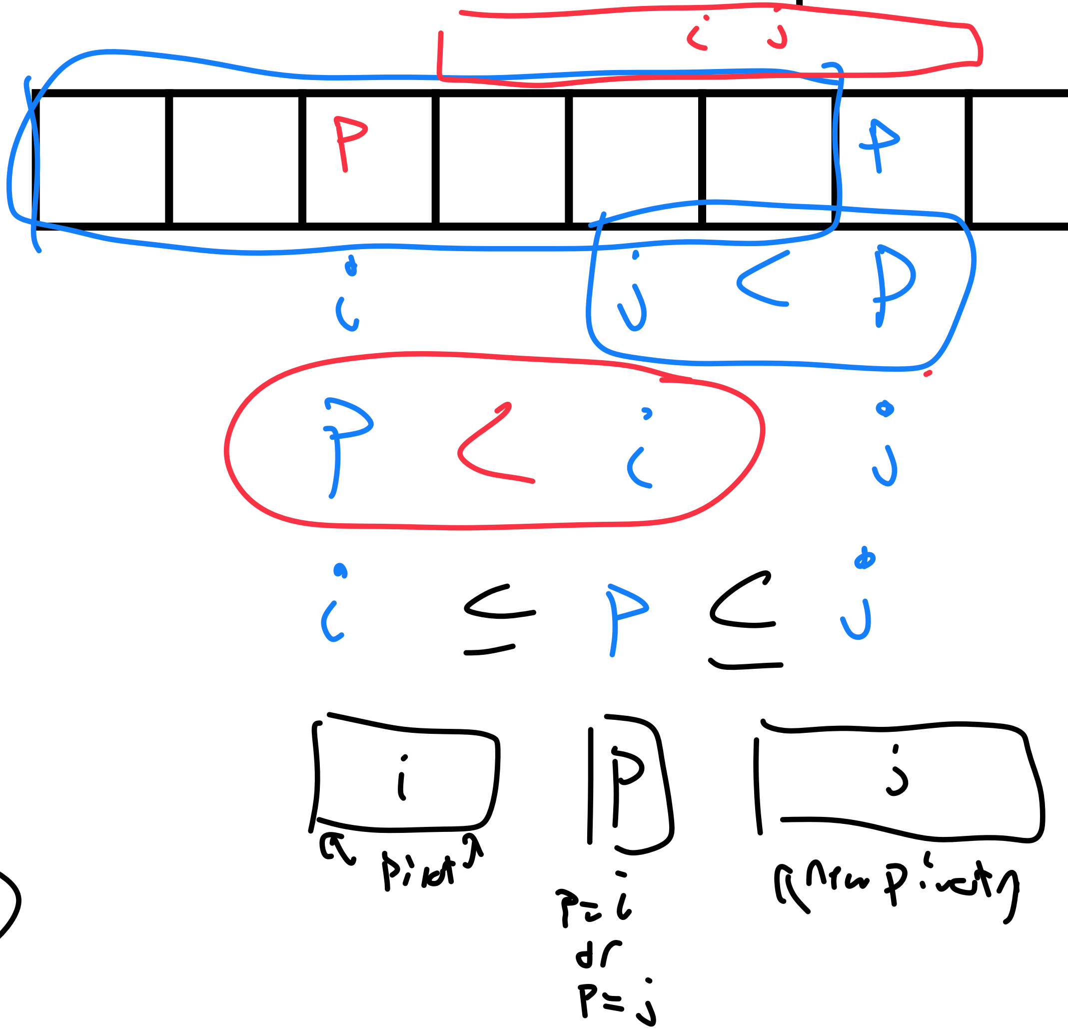
By IH $\dfrac{2}{j-i+1} \cdot \Pr[j<p]$

$+ \Pr[X_{i,j}=1 \mid p<i] \cdot \Pr[p<i]$

By IH $\dfrac{2}{j-i+1} \cdot \Pr[p<i]$

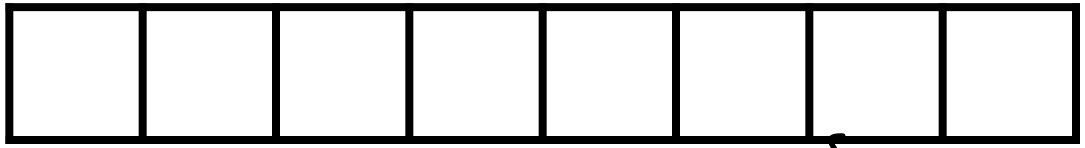
$+ \Pr[X_{i,j}=1 \mid i \le p \le j] \cdot \Pr[i \le p \le j]$

$\dfrac{2}{j-i+1} \cdot \Pr[i \le p \le j]$



Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \dfrac{2}{j - i + 1}$ **Induction:** Assume true for all inputs of $< n$



$$\frac{2}{j-i+1} \cdot \Big(\underbrace{P[j<p] + P[p<i] + P[i \le p \le j]}_{1} \Big)$$

All possible events/states!

i j p

p i j

i p j

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j - i + 1}$$

X is total comparisons

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{\underline{j=i+1}}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n} 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-i+1}\right)$$

$j=i+1$ $\quad i+2$ $\quad \ldots$

Harmonic #

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$

$H_n = \Theta(\log n)$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j-i+1}$$

1+?

$$E[X] = \sum_{i=1}^{n} 2\underbrace{\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-i+1}\right)}$$

$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1) \leq 2n \cdot H_n \leq 2n\ ln\ n$$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \qquad E[X_{ij}] = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n} 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-i+1}\right)$$

(1) Expand out inner sum

$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1)$$

(2) $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$

$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1) \leq 2n \cdot H_n \leq \underline{2n\ ln\ n}$$

(3) $H_n = \theta(log\ n)$

Expectation Analysis: Randomized Quicksort

Summary: Randomized quick sort is $O(n \ log \ n)$ regardless of input

Randomness: choice of pivot

Assumptions: None on data / distribution

↳ can we get a real random #?

Randomization in Algorithms

1. Assume input data is random to estimate average-case performance

2. Use randomness inside algorithm to estimate expected running time

100% accurate but maybe slow

Randomized quicksort $\quad O(n^2)$

↳ Avg case is good!

3. Use randomness inside algorithm to approximate solution in fixed time

↳ Not 100% accurate & fast

↳ Fermat's primality test

Probabilistic Accuracy: Fermat primality test

Pick a random a in the range $[2, \ p - 2]$

If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$

But… ***sometimes*** if n is composite and $a^{n-1} \equiv 1 \pmod{n}$

Probabilistic Accuracy: Fermat primality test

	$a^{p-1} \equiv 1 \pmod{p}$	$a^{p-1} \not\equiv 1 \pmod{p}$
p is prime		
p is not prime		

Probabilistic Accuracy: Fermat primality test

Let's assume $\alpha = .5$

First trial: $a = a_0$ and prime test returns 'prime!'

Second trial: $a = a_1$ and prime test returns 'prime!'

Third trial: $a = a_2$ and prime test returns 'not prime!'

Is our number prime?

What is our **false positive** probability? Our **false negative** probability?

Probabilistic Accuracy: Fermat primality test

Summary: Randomized algorithms can also have fixed (or bounded) runtimes at the cost of probabilistic accuracy.

Randomness:

Assumptions:

Types of randomized algorithms

A **Las Vegas** algorithm is a randomized algorithm which will always give correct answer if run enough times but has no fixed runtime.

A **Monte Carlo** algorithm is a randomized algorithm which will run a fixed number of iterations and may give the correct answer.

Next Class: Randomized Data Structures

Sometimes a data structure can be **too ordered / too structured**

Randomized data structures rely on **expected** performance

Randomized data structures 'cheat' tradeoffs!