Data Structures
MST 2

CS 225
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November 1, 2024

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Department of Computer Science

Learning Objectives

Review the minimum spanning tree (with weights)

Review Kruskal’s / Prim’s MST Algorithms

Focus on determining Big O of complex pseudocode

Compare implementations under different conditions

Summary: DFS and BFS

$|V| = n, |E| = m$

Both are **O(n+m)** traversals! They label every edge and every node

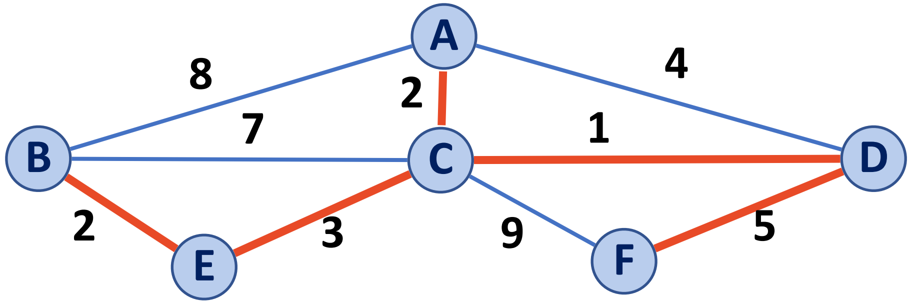
BFS	**DFS**
Solves unweighted MST	Solves unweighted MST
Solves shortest path	
Solves cycle detection	Solves cycle detection
Memory bounded by width	Memory bounded by longest path

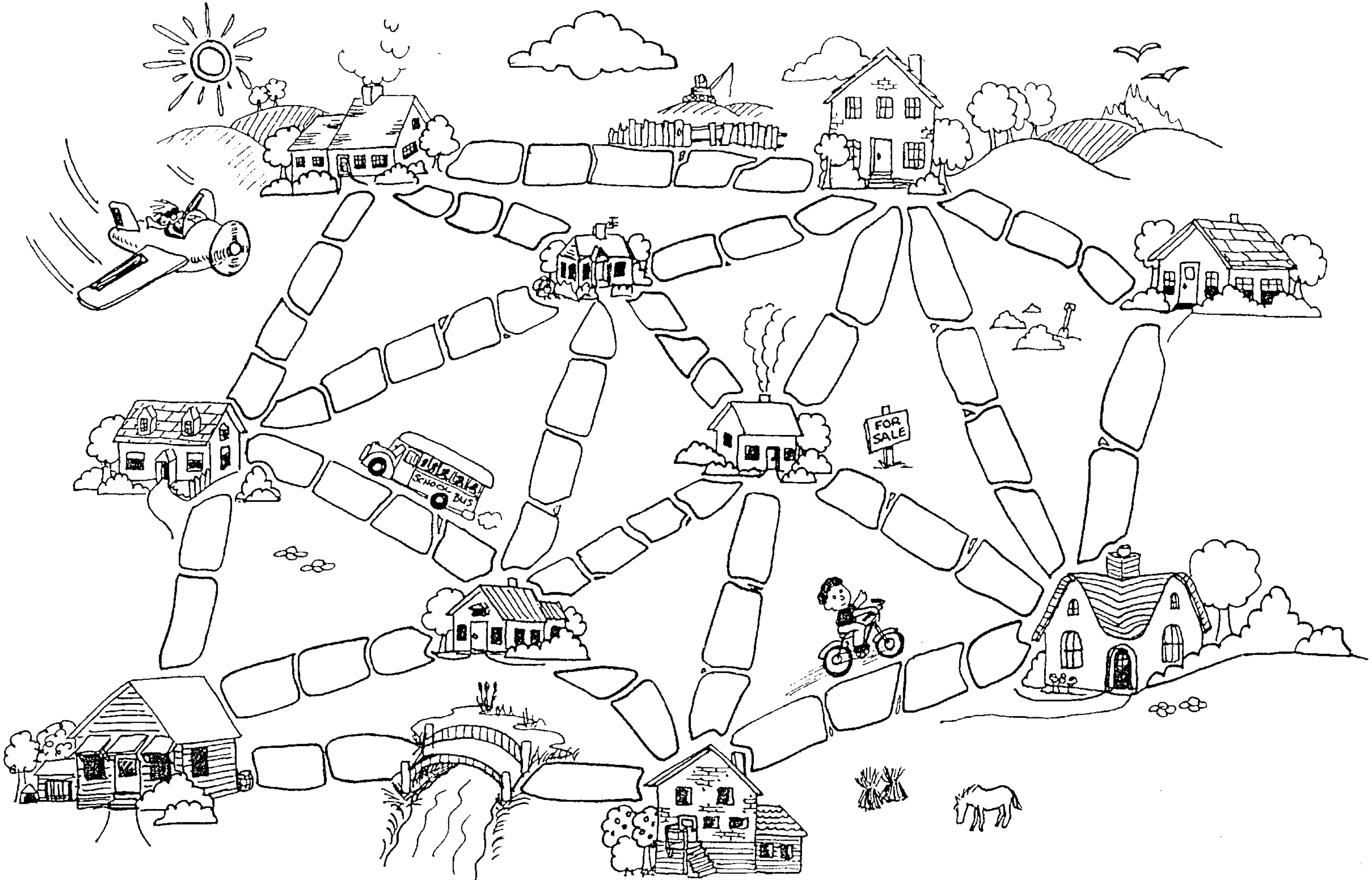
Minimum Spanning Tree Algorithms

Input: Connected, undirected graph **G** with edge weights (unconstrained, but must be additive)

Output: A graph G’ with the following properties:

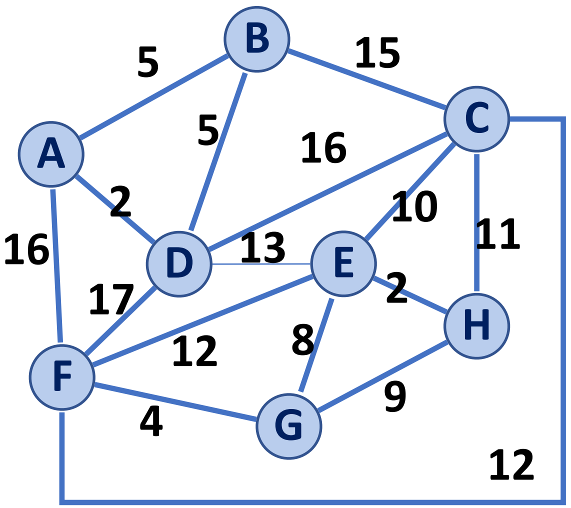
- G’ is a spanning graph of G
- G’ is a tree (connected, acyclic)
- G’ has a minimal total weight among all spanning trees

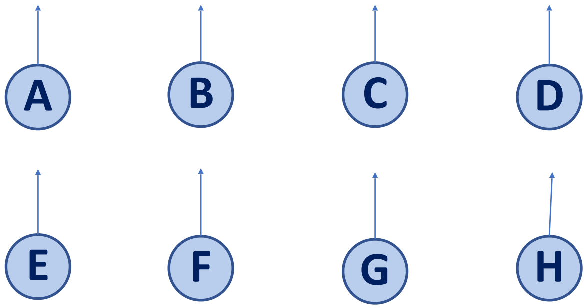



FOR SALE
SCHOOL BUS

Kruskal's Algorithm

(A, D)
(E, H)
(F, G)
(A, B)
(B, D)
(G, E)
(G, H)
(E, C)
(C, H)
(E, F)
(F, C)
(D, E)
(B, C)
(C, D)
(A, F)
(D, F)





1) Build a **priority queue** on edges

2) Build a **disjoint set** on vertices

3) Repeatedly find min edge

If edge connects two sets

Union and record edge

4) Stop after n-1 edges recorded

Kruskal's Algorithm

```
KruskalMST(G):
  DisjointSets forest
  foreach (Vertex v : G.vertices()):
    forest.makeSet(v)

  PriorityQueue Q    // min edge weight
  Q.buildFromGraph(G.edges())

  Graph T = (V, {})

  while |T.edges()| < n-1:
    Vertex (u, v) = Q.removeMin()
    if forest.find(u) != forest.find(v):
       T.addEdge(u, v)
       forest.union( forest.find(u),
                     forest.find(v) )

  return T
```

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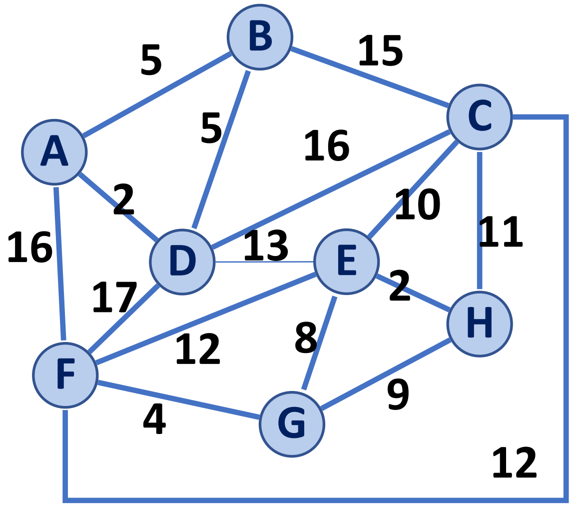
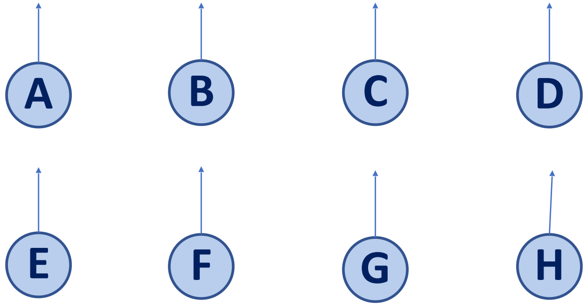
(A, F)
(D, F)



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```

Kruskal's Algorithm

Priority Queue:	Heap	Sorted Array
Building :7		
Each removeMin :12		

```
1  KruskalMST(G):
2    DisjointSets forest
3    foreach (Vertex v : G.vertices()):
4      forest.makeSet(v)
5
6    PriorityQueue Q    // min edge weight
7    Q.buildFromGraph(G.edges())
8
9    Graph T = (V, {})
10
11   while |T.edges()| < n-1:
12     Vertex (u, v) = Q.removeMin()
13     if forest.find(u) != forest.find(v):
14        T.addEdge(u, v)
15        forest.union( forest.find(u),
16                      forest.find(v) )
17
18   return T
19
```

Kruskal's Algorithm

Priority Queue:	Total Running Time
Heap	
Sorted Array	

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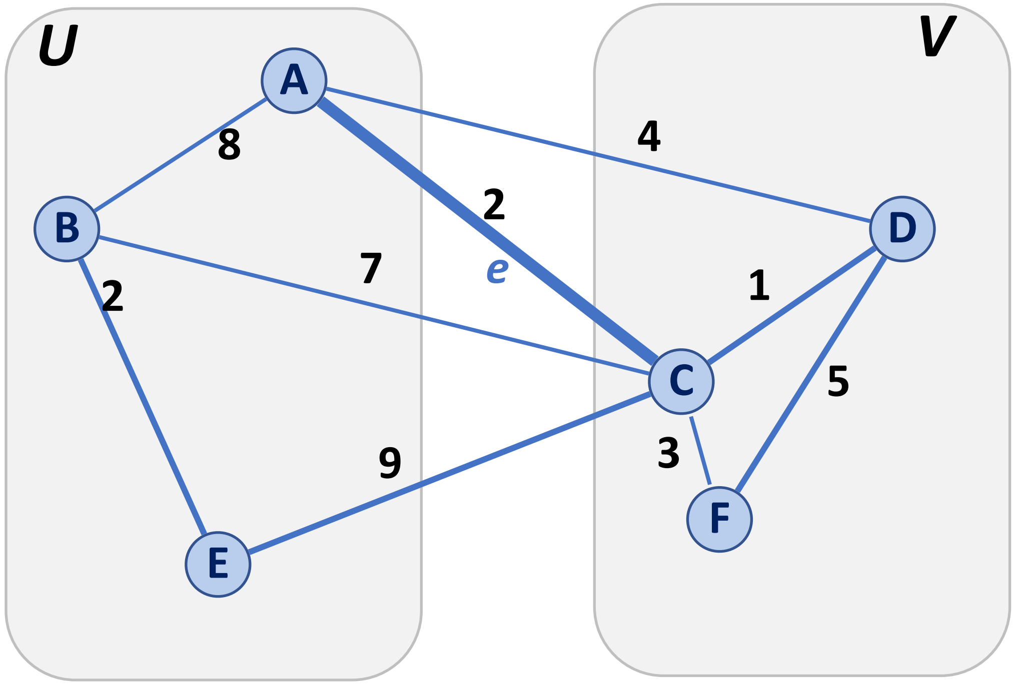
  return T
```

Partition Property

Consider an arbitrary partition of the vertices on **G** into two subsets **U** and **V**.

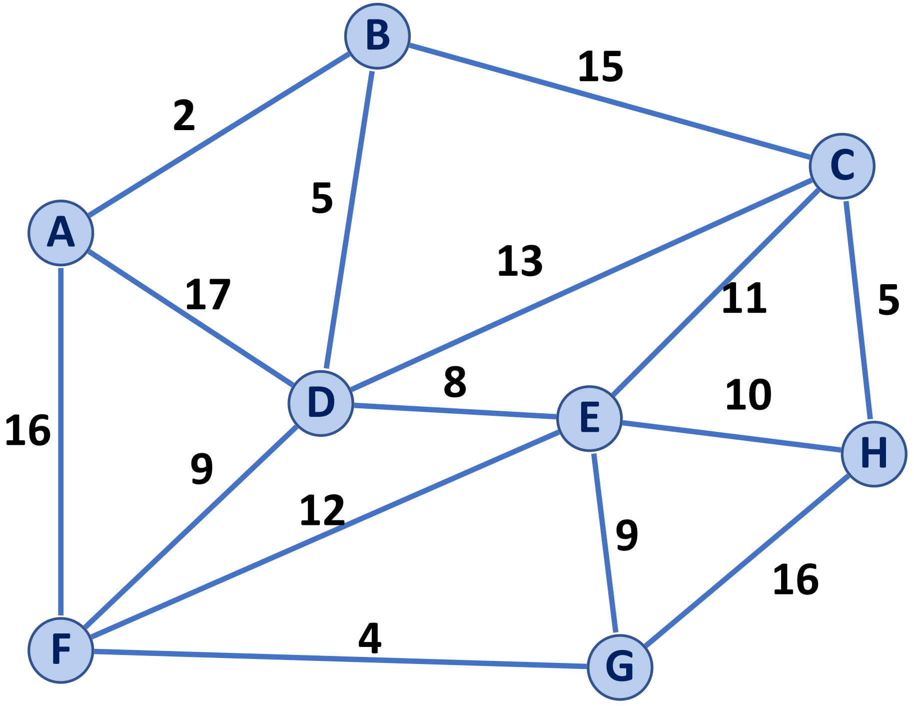
Let **e** be an edge of minimum weight across the partition.

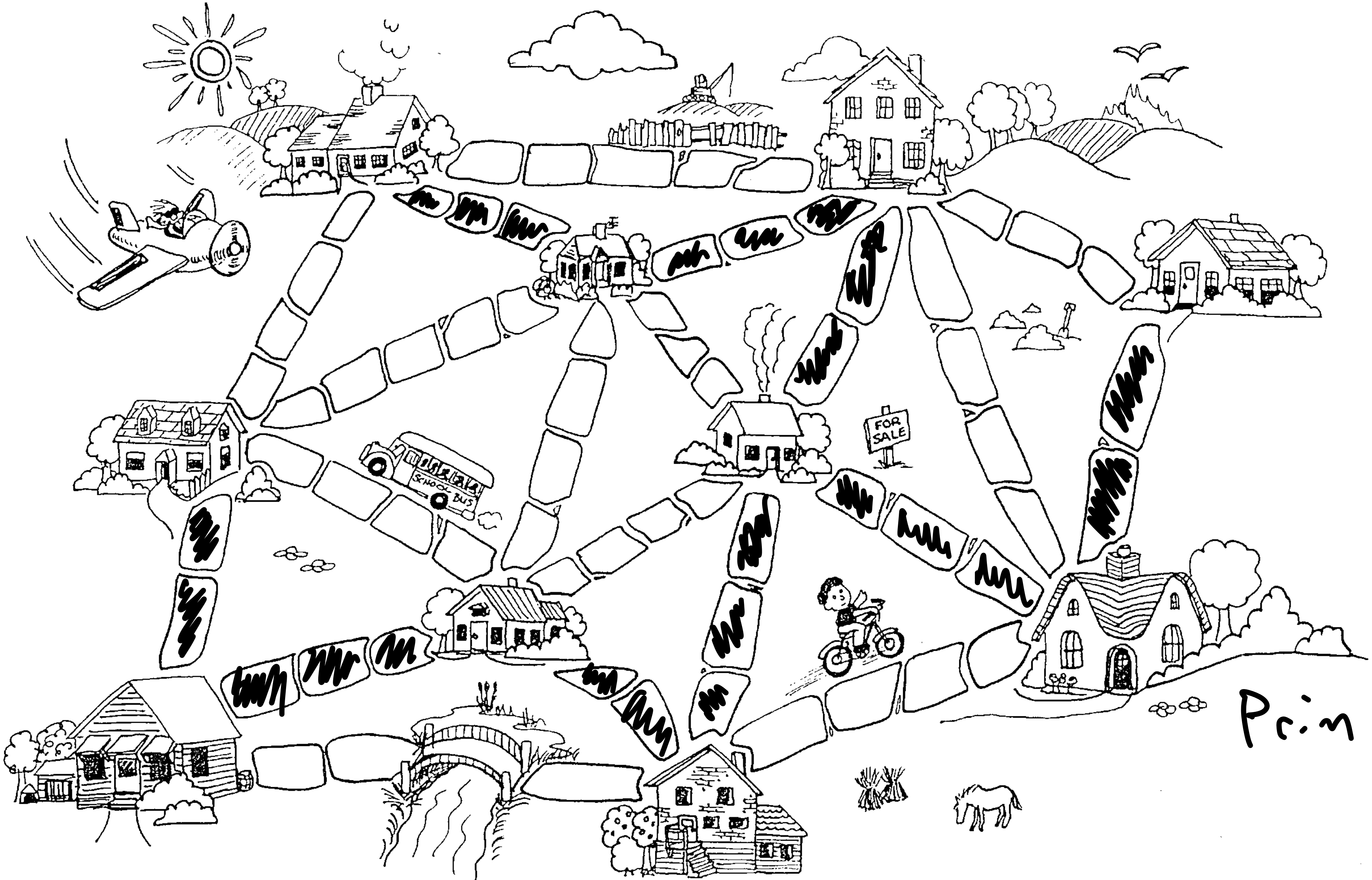
Then **e** is part of some minimum spanning tree.



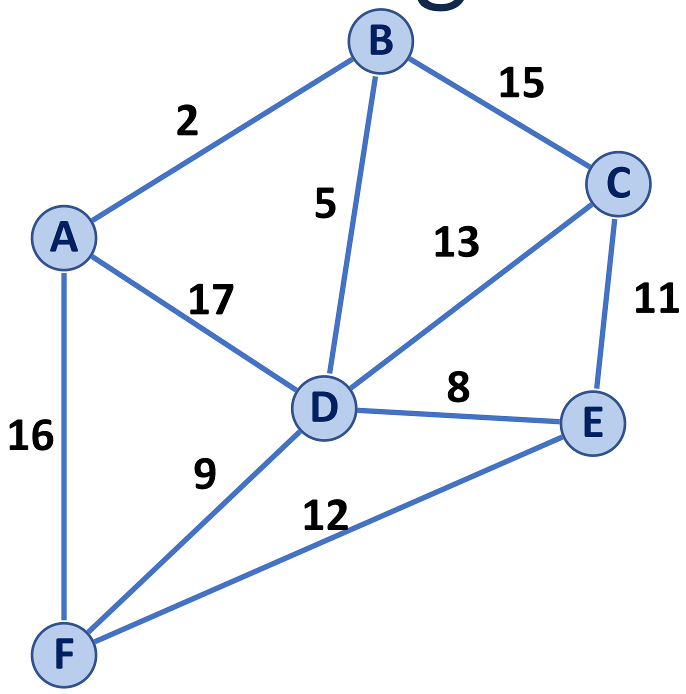
Partition Property

The partition property suggests an algorithm:




SCHOOL BUS
FOR SALE
Prim

Prim’s Algorithm



A	B	C	D	E	F

```
PrimMST(G, s):
  Input: G, Graph;
         s, vertex in G, starting vertex
  Output: T, a minimum spanning tree (MST) of G

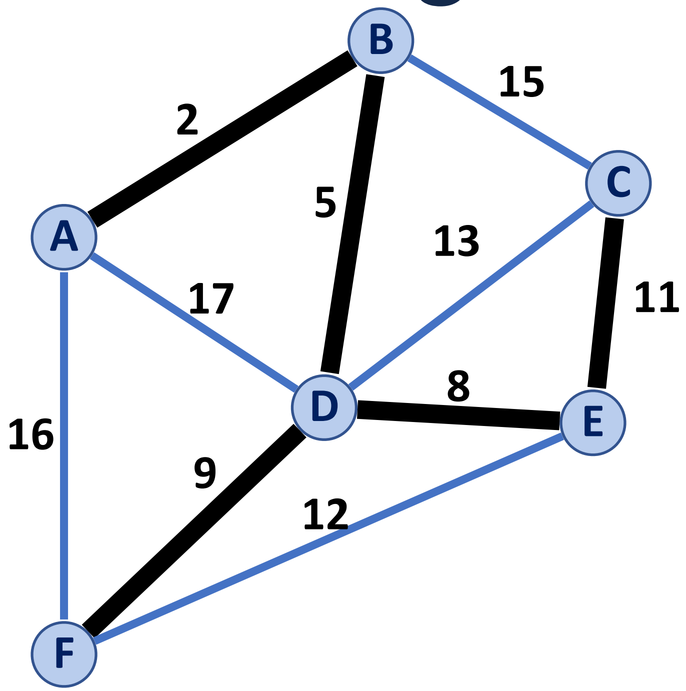
  foreach (Vertex v : G.vertices()):
    d[v] = +inf
    p[v] = NULL
  d[s] = 0

  PriorityQueue Q   // min distance, defined by d[v]
  Q.buildHeap(G.vertices())
  Graph T           // "labeled set"

  repeat n times:
    Vertex m = Q.removeMin()
    T.add(m)
    foreach (Vertex v : neighbors of m not in T):
      if cost(v, m) < d[v]:
        d[v] = cost(v, m)
        p[v] = m

  return T
```

Prim's Algorithm



A	B	C	D	E	F
0, —	2, A	11, E	5, B	8, D	9, D

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```

Prim's Big O

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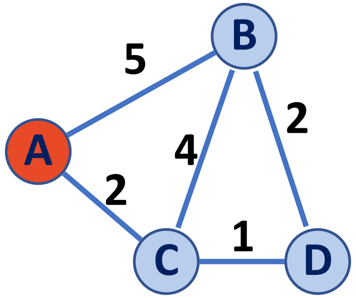
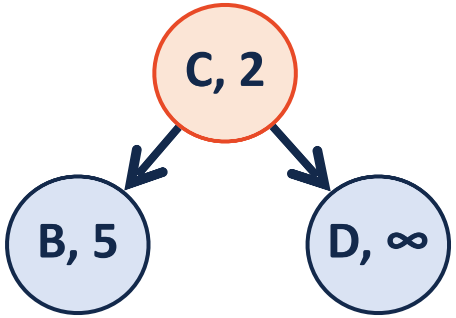
A	B	C	D
0	5	2	∞



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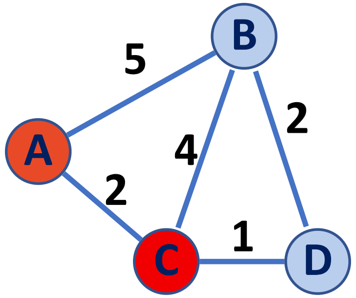
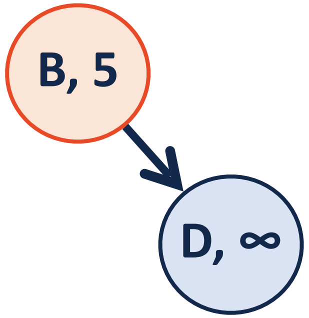
```

	Adj. Matrix	Adj. List
Heap	O(n) + ___________ + O(n^2) + ________________	O(n) + ___________ + O(m) + ________________

A	B	C	D
0	5	2, A	∞



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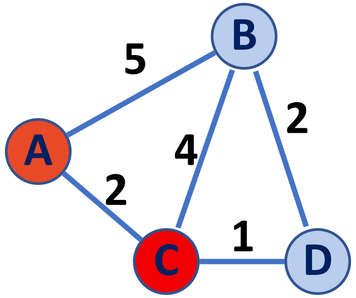
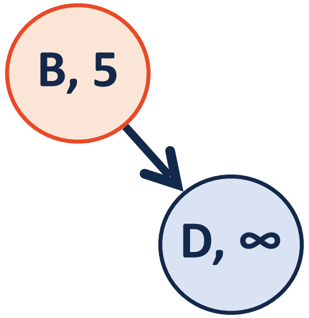
```

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + ________________	O(n) + O(n log n) + O(m) + ________________

A	B	C	D
0	5	2, A	∞



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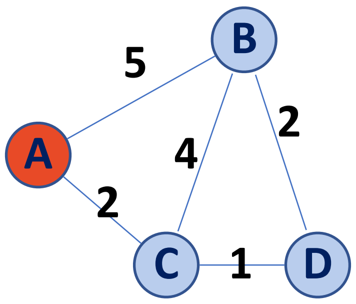
```

1) Change minheap value

2) HeapifyUp()

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + O(m log n)	O(n) + O(n log n) + O(m) + O(m log n)

(A, 0)
(D, ∞)
(C, 2)
(B, 5)

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```

	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$
Unsorted Array		

Prim's Algorithm

Sparse Graph:

Dense Graph:

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PrimMST(G, s):
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	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$
Unsorted Array	$O(n^2)$	$O(n^2)$

MST Algorithm Runtime:

Kruskal's Algorithm:

$O(n + m \log(n))$

Prim's Algorithm:

$O(n \log(n) + m \log(n))$

Sparse Graph:

Dense Graph:

Suppose I have a new heap:

	Binary Heap	Fibonacci Heap
Remove Min	O(lg(n))	O(lg(n))
Decrease Key	O(lg(n))	O(1)*

What's the updated running time?

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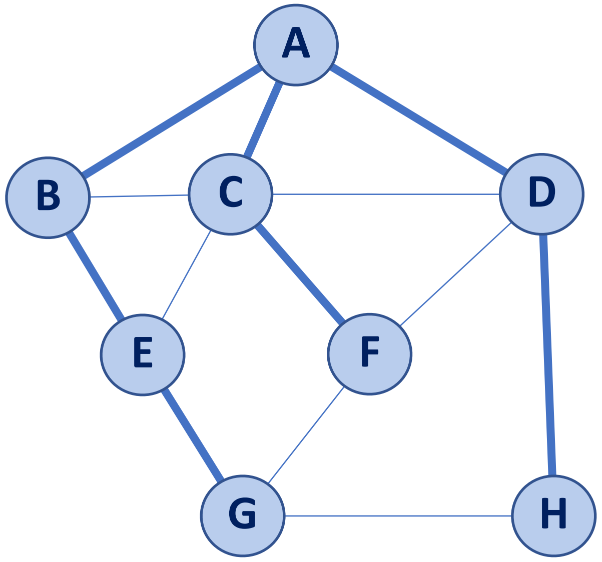
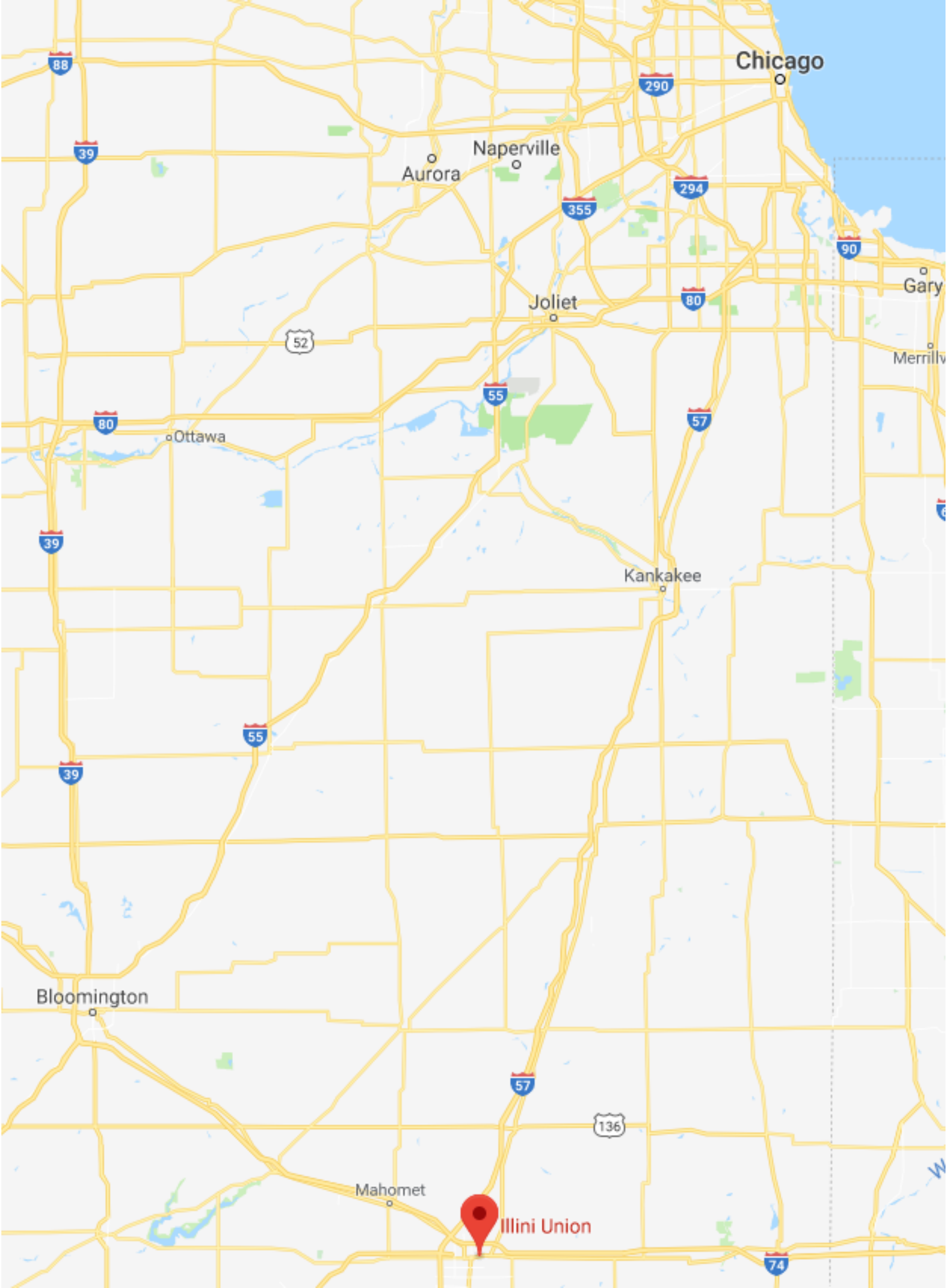
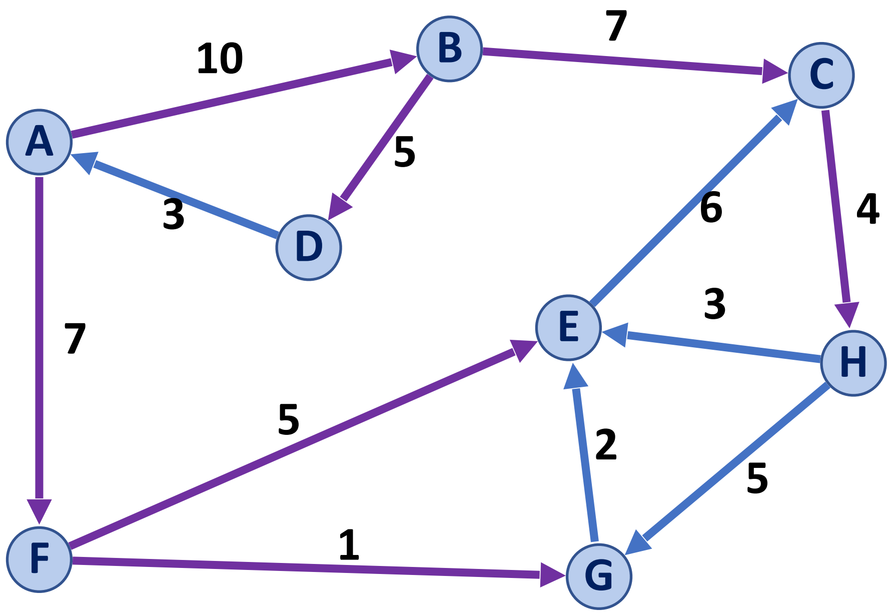
Shortest Path





Dijkstra's Algorithm (SSSP)

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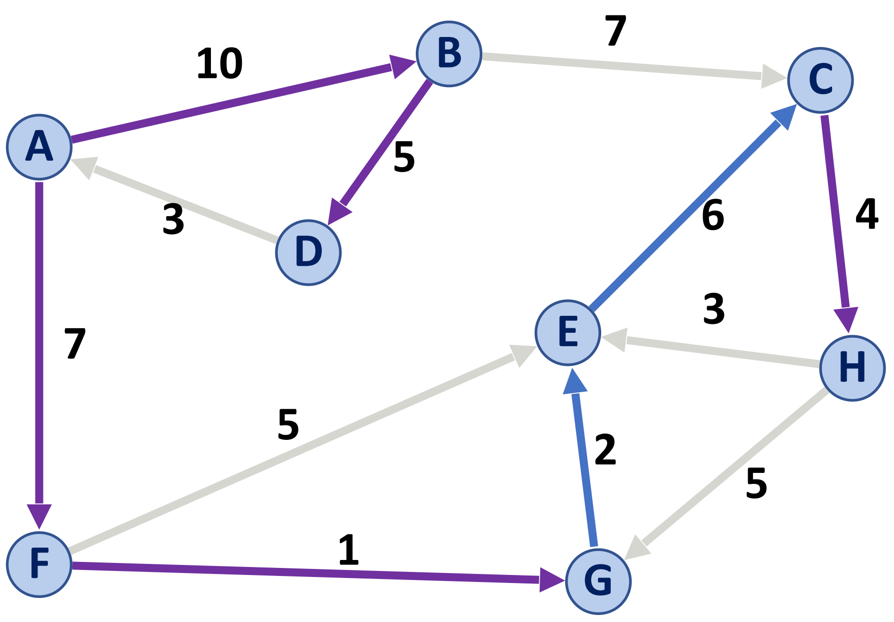
      if ________________ < d[v]:
        d[v] = ________________
        p[v] = u
```

A	B	C	D	E	F	G	H
--							
0							

Dijkstra’s Algorithm (SSSP)

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```

A	B	C	D	E	F	G	H
--	A	E	B	G	A	F	C
0	10	16	15	10	7	8	20

Dijkstra's Algorithm (SSSP)

What is the running time of Dijkstra's Algorithm?

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