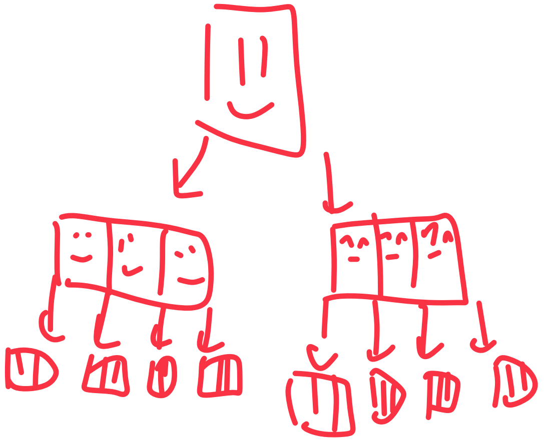
Data Structures
BTree Analysis

CS 225
Brad Solomon

October 9, 2024



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Department of Computer Science

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RESEARCH 101

DIVE INTO THE MYSTERIES OF RESEARCH OPPORTUNITIES AT SIEBEL SCHOOL AND DISCOVER HOW YOU CAN GET INVOLVED! PRESENTED BY PROFESSOR BRAD SOLOMON W/ STUDENT Q & A PANEL.

October 10th, 1-2:30 PM, Siebel 2405

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More questions? Contact

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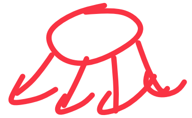
Learning Objectives

→ Finish implementation

Discuss the importance of M in a B Tree

Analyze the performance of the B Tree

BTree Properties



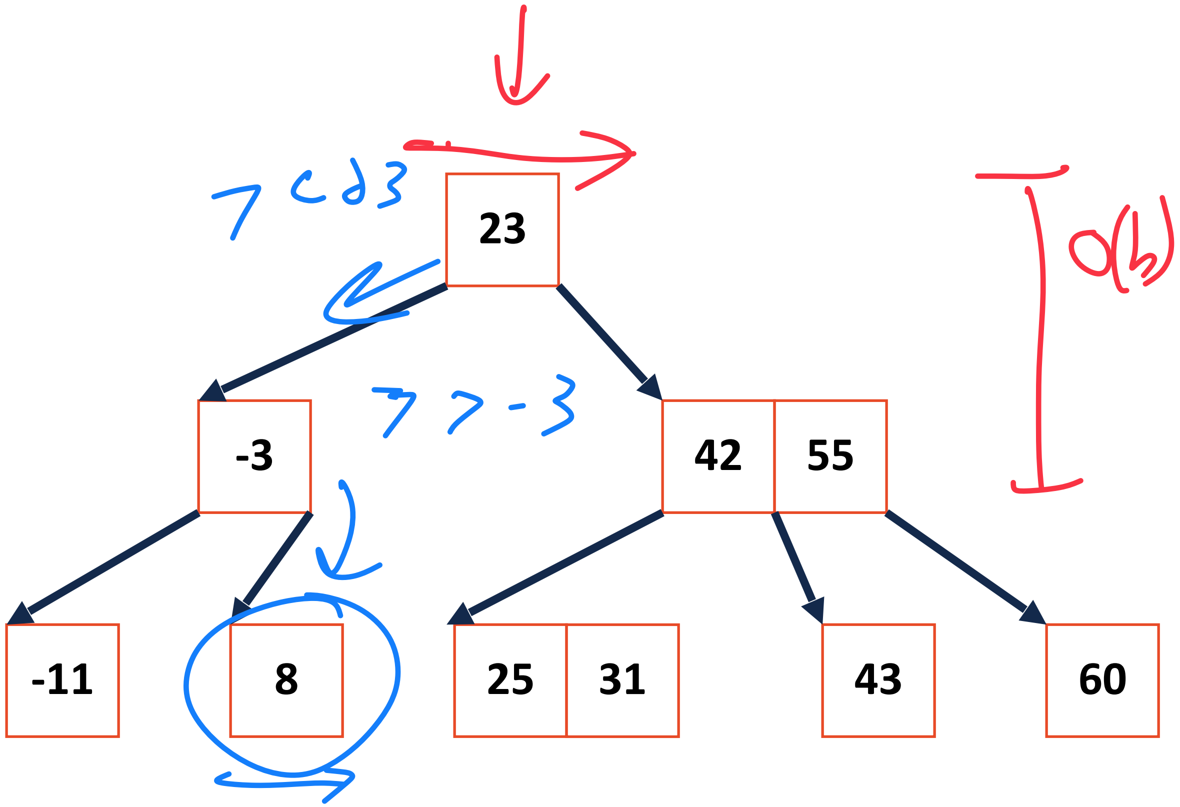
up to m children

A **BTrees** of order **m** is an m-ary tree and by definition:

- All keys within a node are ordered
- All nodes contain no more than **m-1** keys.
- All internal nodes have exactly **one more child than keys**
- All leaves in the tree are at the same level.

BTree Find

Find(7)



Base Case:

If root is empty, return

If leaf, do array find() and return

Recursive Step:

Array find() for match or first greater value

Recurse on appropriate child

Tip: Index of first greater value is index of child we want to visit!

BTree Insertion

M = 5

sorted

Given the appropriate BTreeNode, insert is array insert

Insert(1)
Insert(2)
Insert(3)
Insert(4)
Insert(5)
Insert(6)
Insert(7)
Insert(8)

1	2	3	4	5

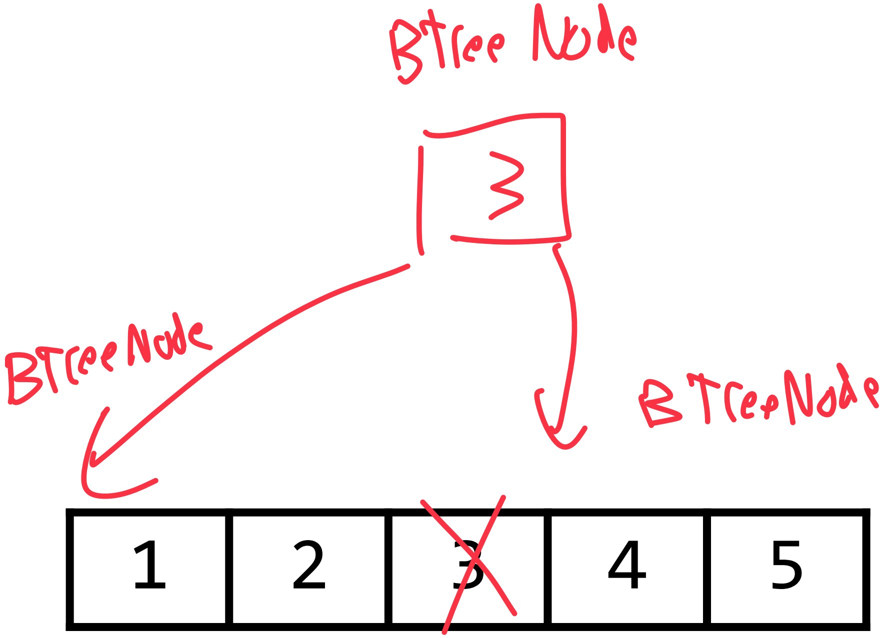
BTree Insertion

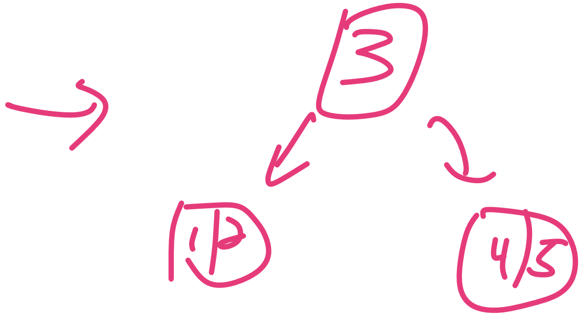
M = 5

When we hit **M** items, split into three nodes!

1) Find median
2) "Raise median up"

↳ Cut array in half as 2 new BTree Nodes





Insert(1)
Insert(2)
Insert(3)
Insert(4)
Insert(5)
Insert(6)
Insert(7)
Insert(8)

M-1 items
MAX

BTree Insertion

M = 5

"Given appropriate BTreeNode" == Find()

Insert(1)

Insert(2)

Insert(3)

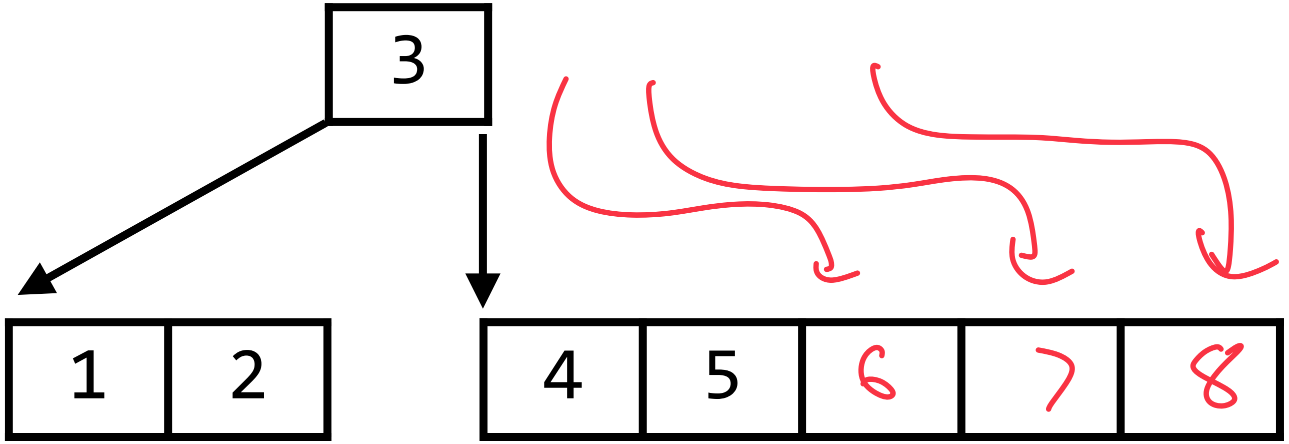
Insert(4)

Insert(5)

Insert(6)

Insert(7)

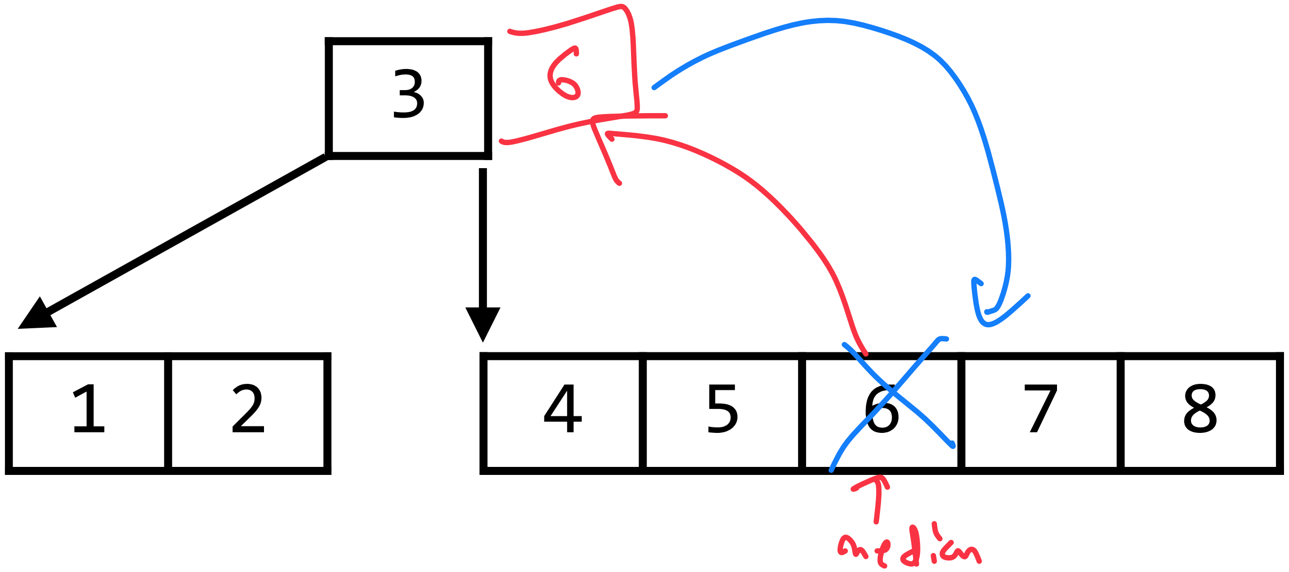
Insert(8)



BTree Insertion

M = 5

If parent node already exists, split adds new key.



Insert(1)

Insert(2)

Insert(3)

Insert(4)

Insert(5)

Insert(6)

Insert(7)

Insert(8)

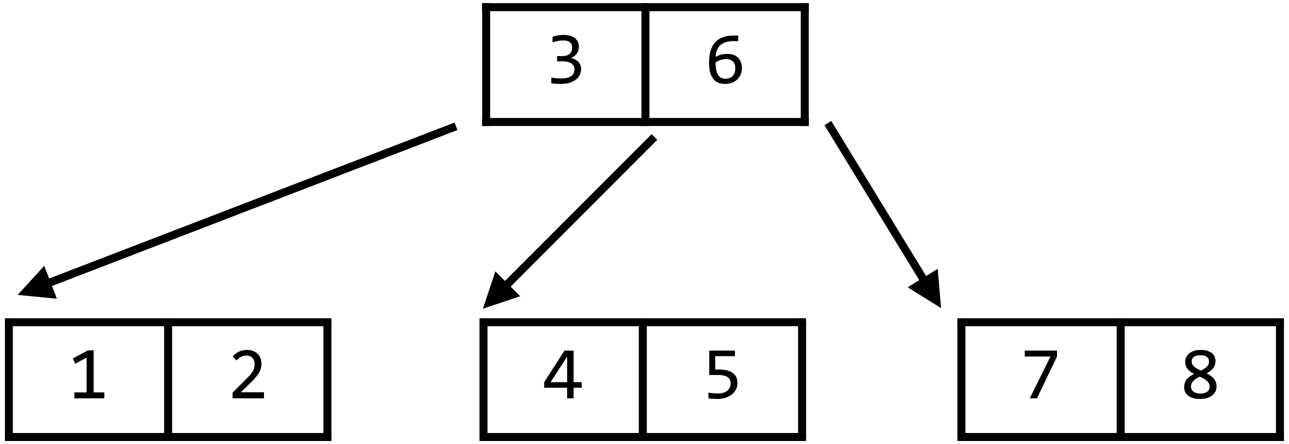
BTree Insertion

min # keys = int$\left(\frac{M}{2}\right)$

M = 5

If parent node already exists, split instead adds new key.

Insert(1)
Insert(2)
Insert(3)
Insert(4)
Insert(5)
Insert(6)
Insert(7)
Insert(8)



BTree Insertion

M = 5

Problem 3: I need to find median value AFTER inserting the **M**th value

10				

Insert(10)

5	10			

Insert(5)

. . .

2	5	7	9	10

Insert(2)

BTree Insertion

M = 5

Problem 3: I need to find median value AFTER inserting the **M**th value

m-1

10				

Insert(10)

5	10			

Insert(5)

. . .

2	5	7	9	10

Insert(2)

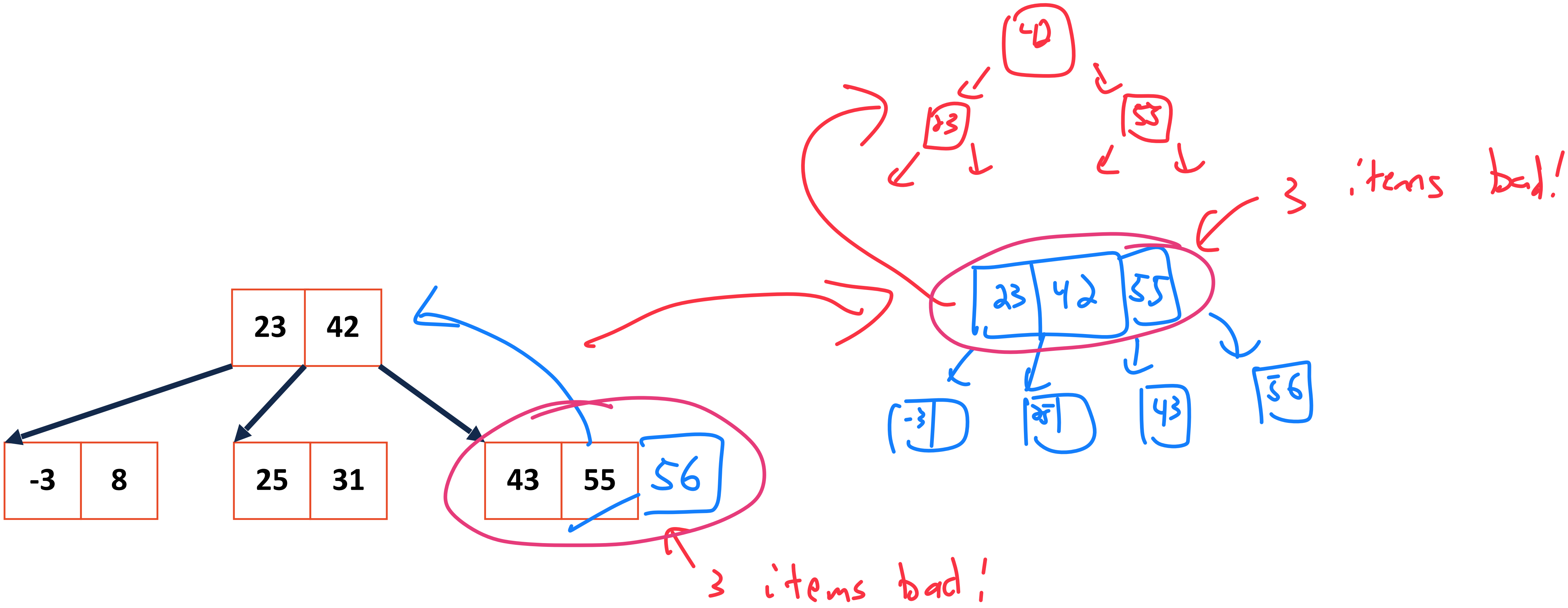
keys :)

Non-Optimal Solution: Pre-allocate **M** size arrays for every node!

BTree Recursive Insert

Insert(56), M = 3

Insert always starts at a leaf but can propagate up repeatedly.



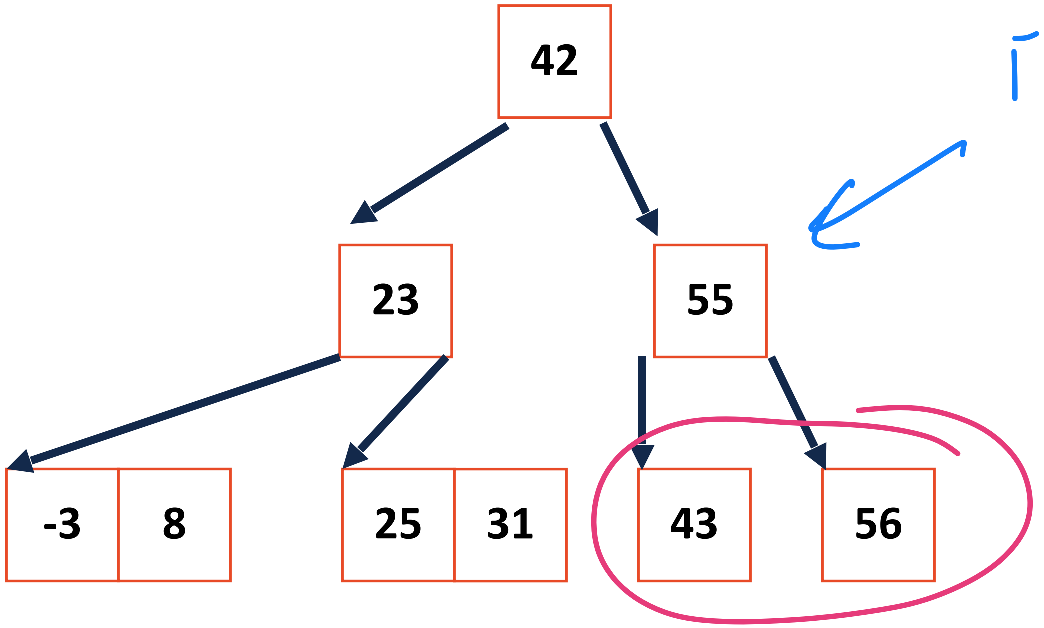
BTree Recursive Insert

Insert(56), M = 3

Insert always starts at a leaf but can propagate up repeatedly.

↳ O(h) times

$\lceil \frac{M}{2} \rceil$ children



why was this split up?

↳ Enforce order

↳ M tells us this is ok!

BTree Remove

BTree removal is complicated! **It won't be part of the lab.**

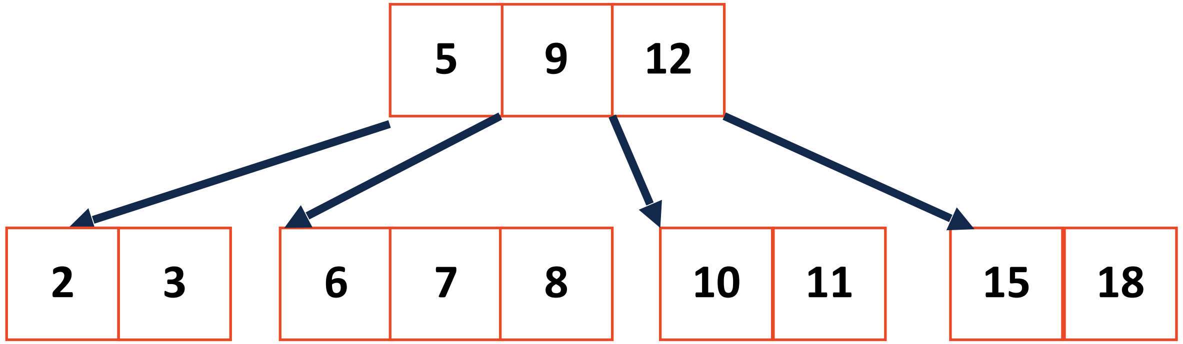
If we have time at the end of the day today we will discuss it

If __

If __

If __

If



Boring?

BTree Node (of order m)

Network packet is ~1500 bytes
↳ Make sure most info is stored!

Brainstorm together: What value of **m** should we be using?

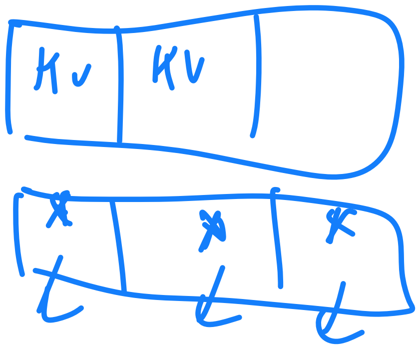
↳ We want efficient memory
↳ Load a node and have many (K,V) pairs

512 bytes in RAM

What is my memory size?
what is size of (K,V) pair

an int is 4 bytes

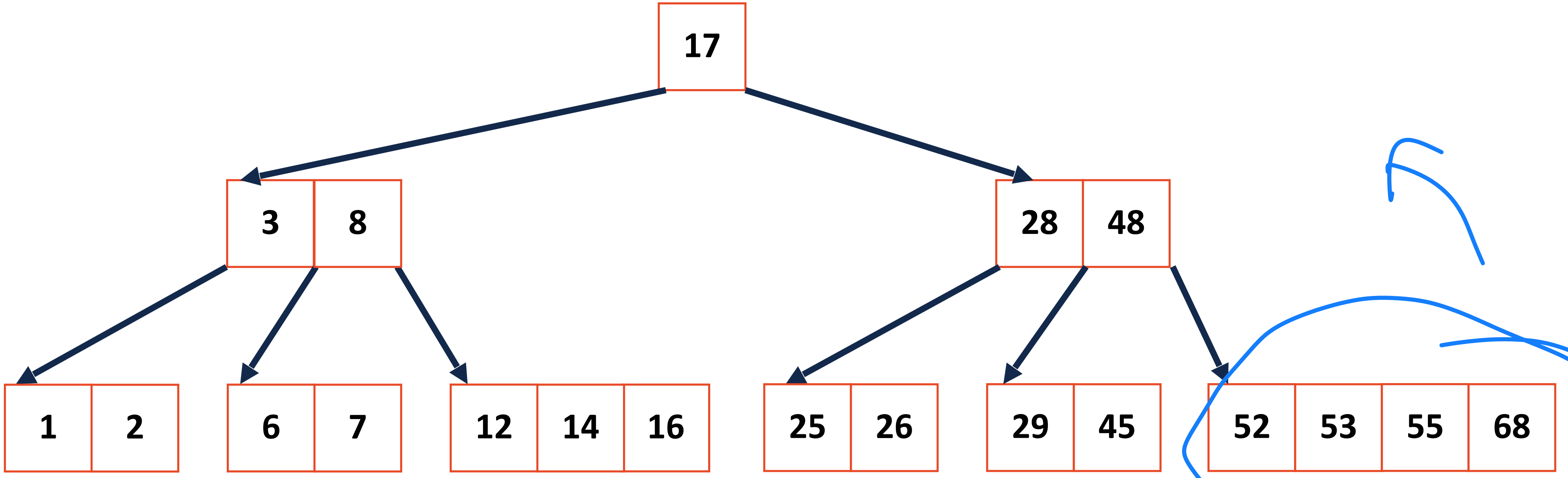
512/4 − 1 − pointer size is "ideal" M



BTree *of Order M*

If I tell you this is a valid BTree, what is the **lower bound** of **M**?

5



4 keys implies

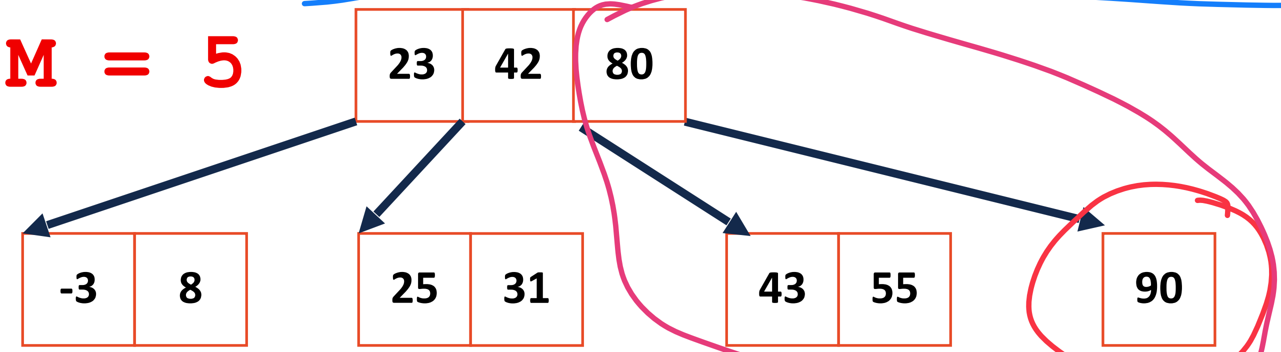
$M \leq 5$

BTree Size Restrictions

M = # children max
$M-1$ is max keys

We have max on ~~nodes~~ Keys, but do we have min? Are these trees valid?

M = 5



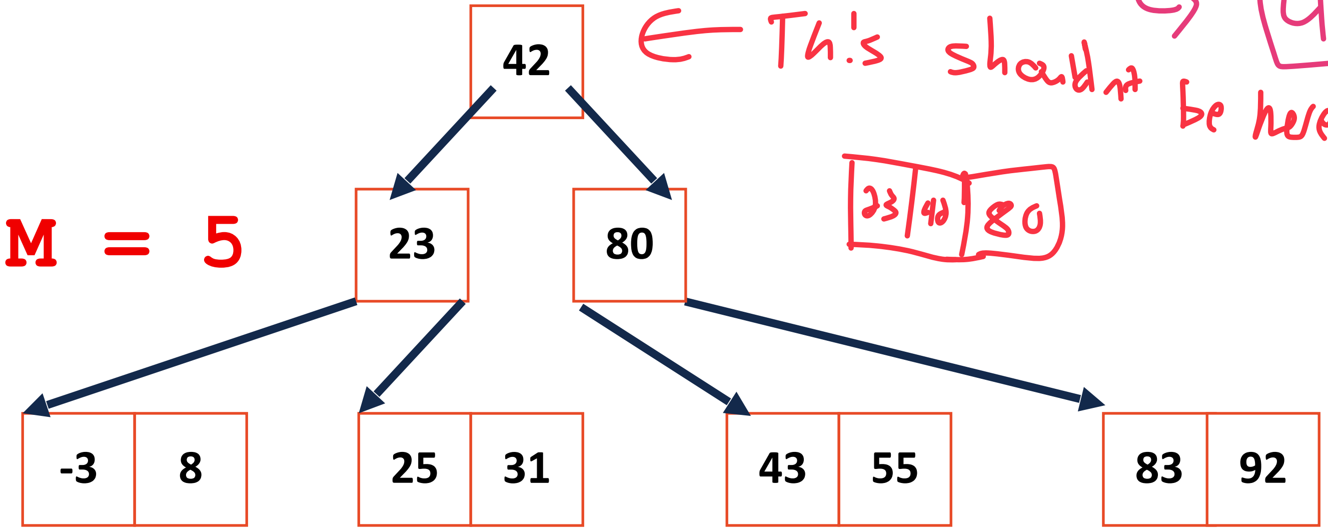
No not valid

← this is below min $\frac{M}{2}$

same idea

43 | 55 | 80 | 90

M = 5



← This shouldn't be here

23 | 42 | 80

Not valid

BTree Properties

A **BTrees** of order **m** is an m-ary tree and by definition:

- All keys within a node are ordered
- All nodes contain no more than **m-1** keys.
- All internal nodes have exactly **one more child than keys**

→ 0 children

Root nodes can be a leaf or have $[2, m]$ children.

If int $\lceil \frac{m}{2} \rceil$ is keys, +1 is children

All non-root, internal nodes have $[\lceil \frac{m}{2} \rceil, m]$ children.

All leaves in the tree are at the same level.

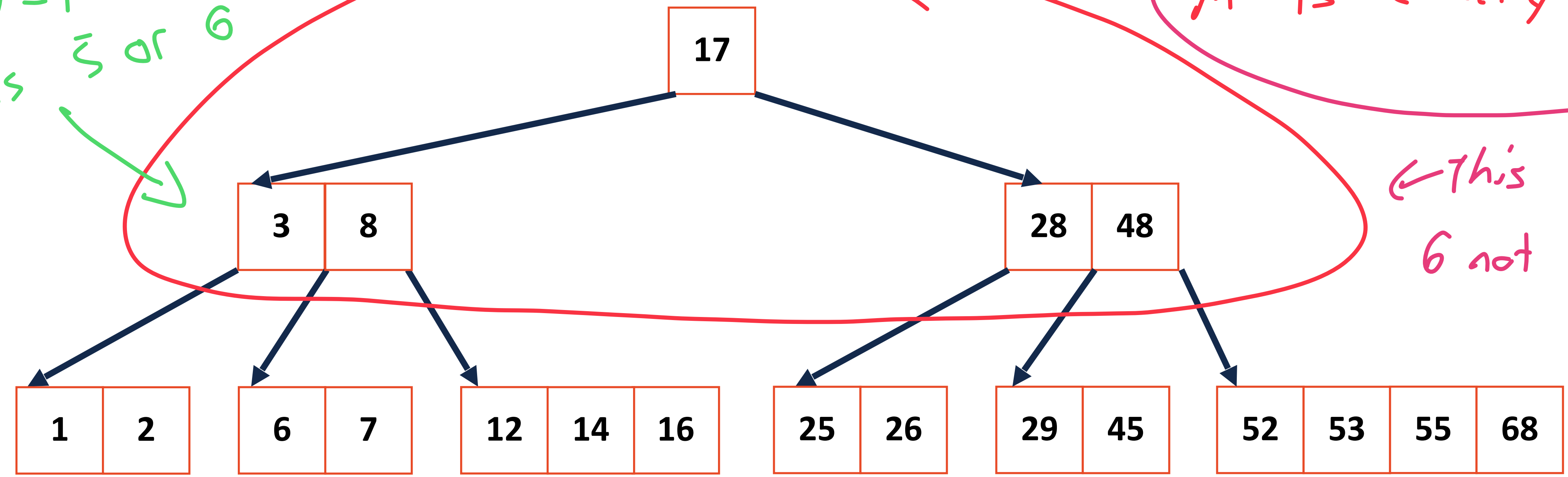
BTree

$M = 5$

If I tell you this is a valid BTree, what is the **precise value** of m?

$\lceil \frac{m}{2} \rceil - 1$ tells us 5 or 6

m is exactly 5



← this says 6 not ok

$m \leq 5$

If $M = 6$, root is $\boxed{3 | 8 | 17 | 28 | 48}$

BTree Analysis

Like the BST, BTree height determines the runtime of our operations!

Claim: The BTree structure limits our height to $O(log_m(n))$ *

Proof: We want to find a relationship for BTrees between the number of keys (**n**) and the height (**h**).

BTree Analysis

Strategy:

We will first count the number of nodes, level by level. 2

Then, we will add the minimum number of keys per node (**n**).

The minimum number of nodes will tell us the largest possible height (**h**), allowing us to find an upper-bound on height.

Key Facts:

2 ↔ m

Root nodes can be a leaf or have **[2, m]** children.

All non-root, internal nodes have **[ceil(m/2), m]** children.

BTree Analysis

All internal nodes have $t = \lceil \frac{m}{2} \rceil$ children

Minimum number of **nodes** for a BTree of order m **at each level:**

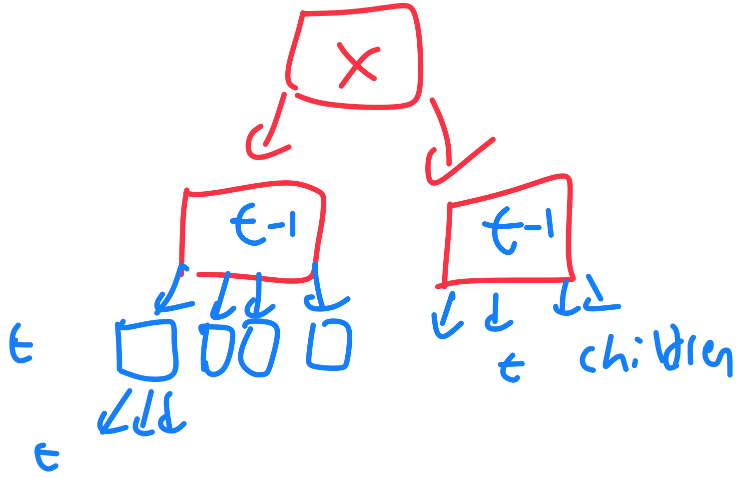
Root: 1

Level 1: 2

Level 2: $2t$

Level 3: $2t^2$

Level h: $2t^{h-1}$



BTree Analysis

$t = \lceil \frac{m}{2} \rceil$

The **min total number of nodes** is the sum of all the levels:

$$1 + 2\sum_{k=0}^{h-1} t^k = 1 + 2\frac{t^h - 1}{t - 1}$$

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}$$

Summation identity

BTree Analysis

$$t = \lceil \frac{m}{2} \rceil$$

The **min total number of nodes:**

$$\text{root} \quad 1 + 2\overbrace{\frac{t^h - 1}{t - 1}}^{\text{internal \& leaf nodes}}$$

The **min total number of keys:**

Root : 1 key / 1 Node

min # keys in internal: $\lceil \frac{m}{2} \rceil - 1 = t - 1$

keys in a leaf: $\lceil \frac{m}{2} \rceil - 1 = t - 1$

every internal & every leaf have min $t - 1$

BTree Analysis

$t = \lceil \frac{m}{2} \rceil$

The **min total number of nodes:** $1 + 2\frac{t^h - 1}{t - 1}$

The **min total number of keys:**

$$= 1 + 2\frac{t^h - 1}{\cancel{t - 1}} * \cancel{(t - 1)}$$

$$= 2t^h - 1$$

Root has how many keys? **1**

Internal nodes? $\lceil \frac{m}{2} \rceil - 1 = t - 1$

Leaf nodes? $\lceil \frac{m}{2} \rceil - 1 = t - 1$

So we can multiply the fraction by $t - 1$

BTree Analysis

$$t = \lceil \frac{m}{2} \rceil$$

The **smallest total number of keys** is: $2t^h - 1$

So an inequality about **n**, the total number of keys:

$n \geq 2t^h - 1$

$n + 1 \geq 2t^h$

$\log_m(n+1) \geq \log_m\left(2^{\lceil \frac{m}{2} \rceil}\right)^h$

Solving for **h**, since **h** is the max number of seek operations:

BTree Analysis

$t = \lceil \frac{m}{2} \rceil$

The **smallest total number of keys** is: $2t^h - 1$

So an inequality about **n**, the total number of keys:

$$n \geq 2t^h - 1$$

+1

$$n + 1 \geq 2t^h$$

Ignored ceil & drop constants

$$log_m\ (n+1) \geq log_m\ \left(2\lceil \frac{m}{2} \rceil^h\right) = log_m\ (m^h) = h$$

Solving for **h**, since **h** is the max number of seek operations:

$$h = O(log_m\ n)$$

BTree Analysis

This is very powerful!

As long as I am *at least* minimally sized, we are $O(\log_m n)$!

BTree Analysis

Given **m=101**, a tree of height **h=4** has:

Minimum Keys:

Maximum Keys:

BTree

The BTree is still used heavily today!

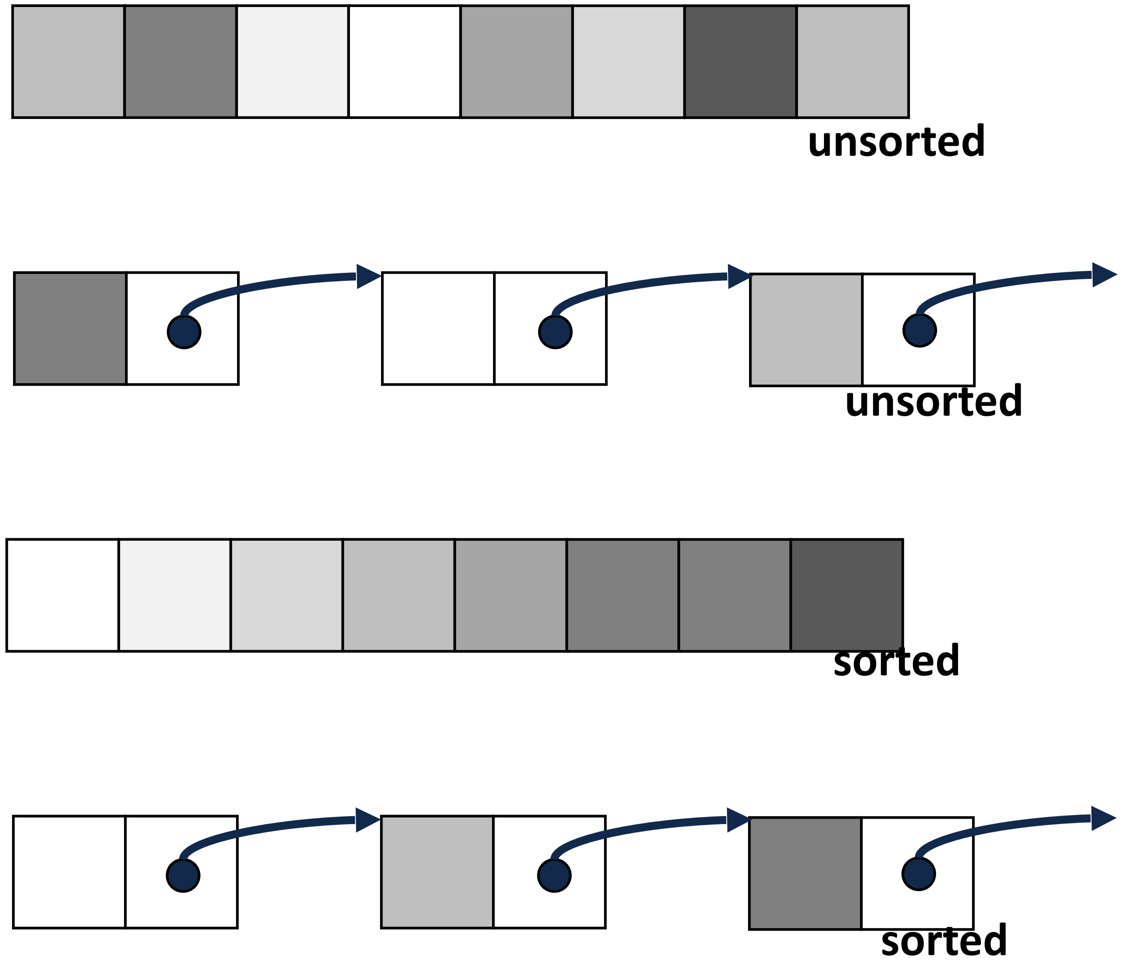
Improvements such as B+Tree and B*Tree exist far outside class scope

Thinking conceptually: Sorting a queue

How might we build a ‘queue’ in which our front element is the min?

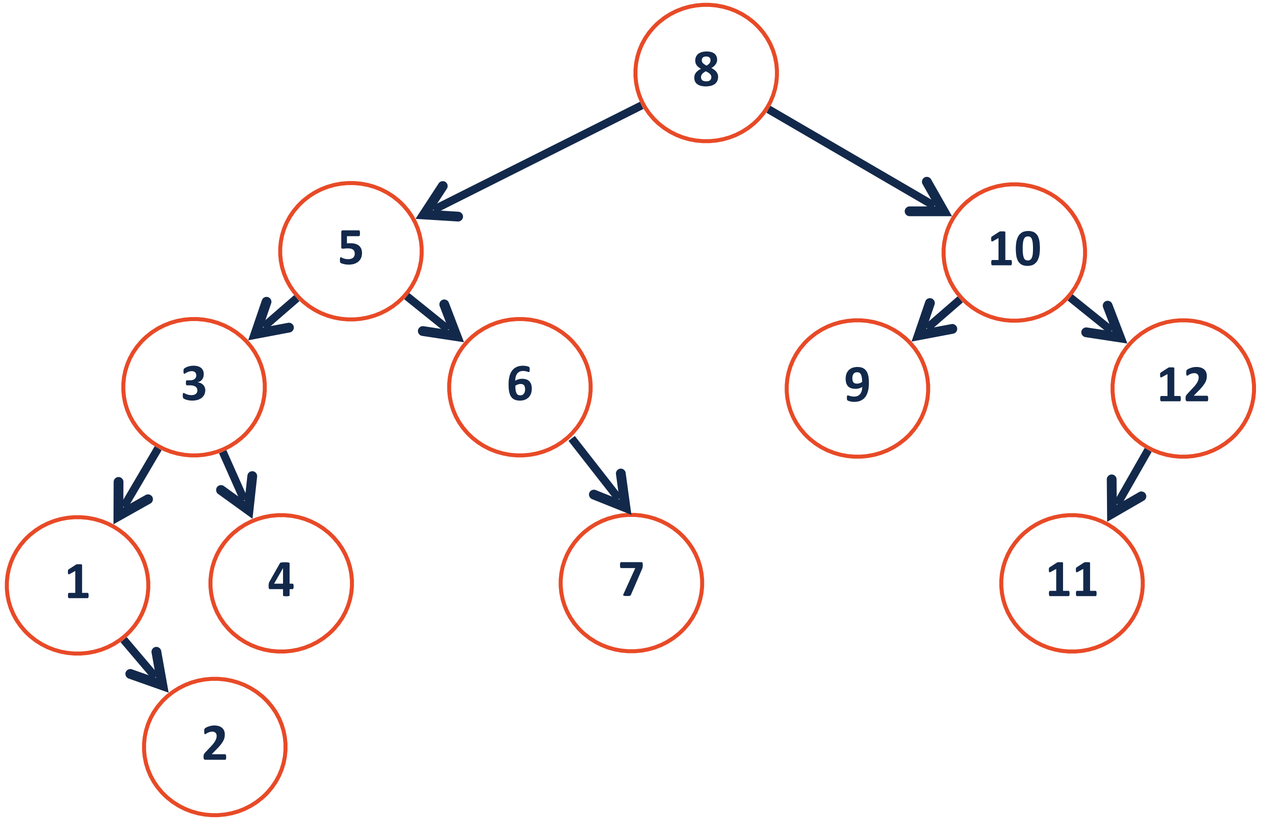
Priority Queue Implementation

insert	removeMin
O(n)	O(n)
O(1)	O(n)
O(n)	O(1)
O(n)	O(1)



Priority Queue Implementation

insert	removeMin



Thinking conceptually: A tree without pointers

What class of (non-trivial) trees can we describe without pointers?