Data Structures
AVL Analysis

CS 225
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September 30, 2024

UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN
Department of Computer Science

No MP this week!

We will cover content necessary for mp_mosaics this week

An opportunity to catch up on work

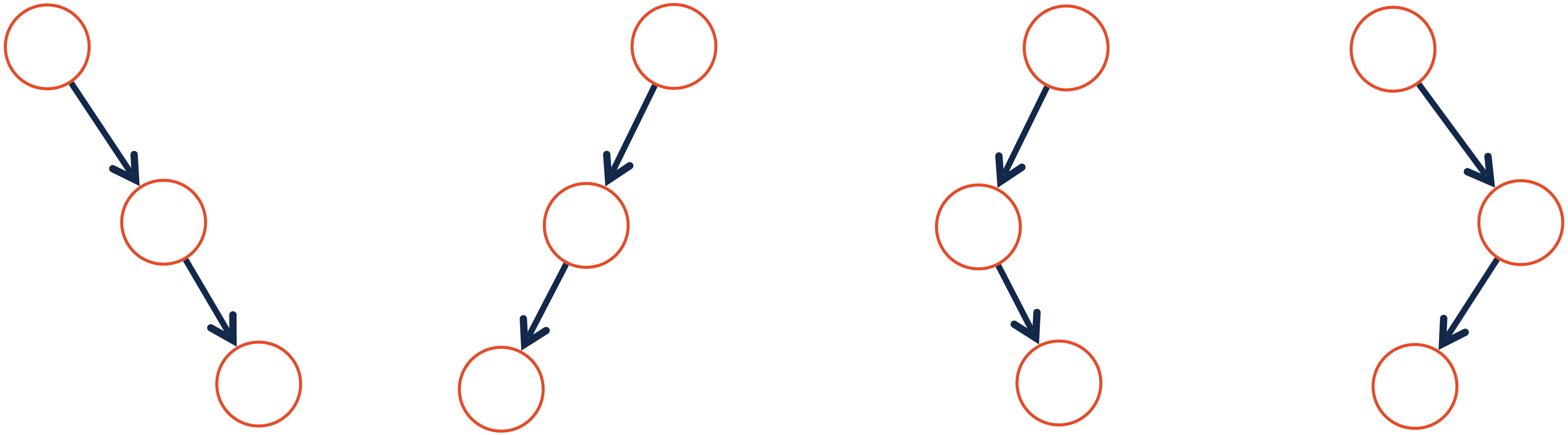
An opportunity to complete the **Informal Early Feedback**

Learning Objectives

Review AVL trees

Prove that the AVL Tree speeds up all operations

AVL Rotations

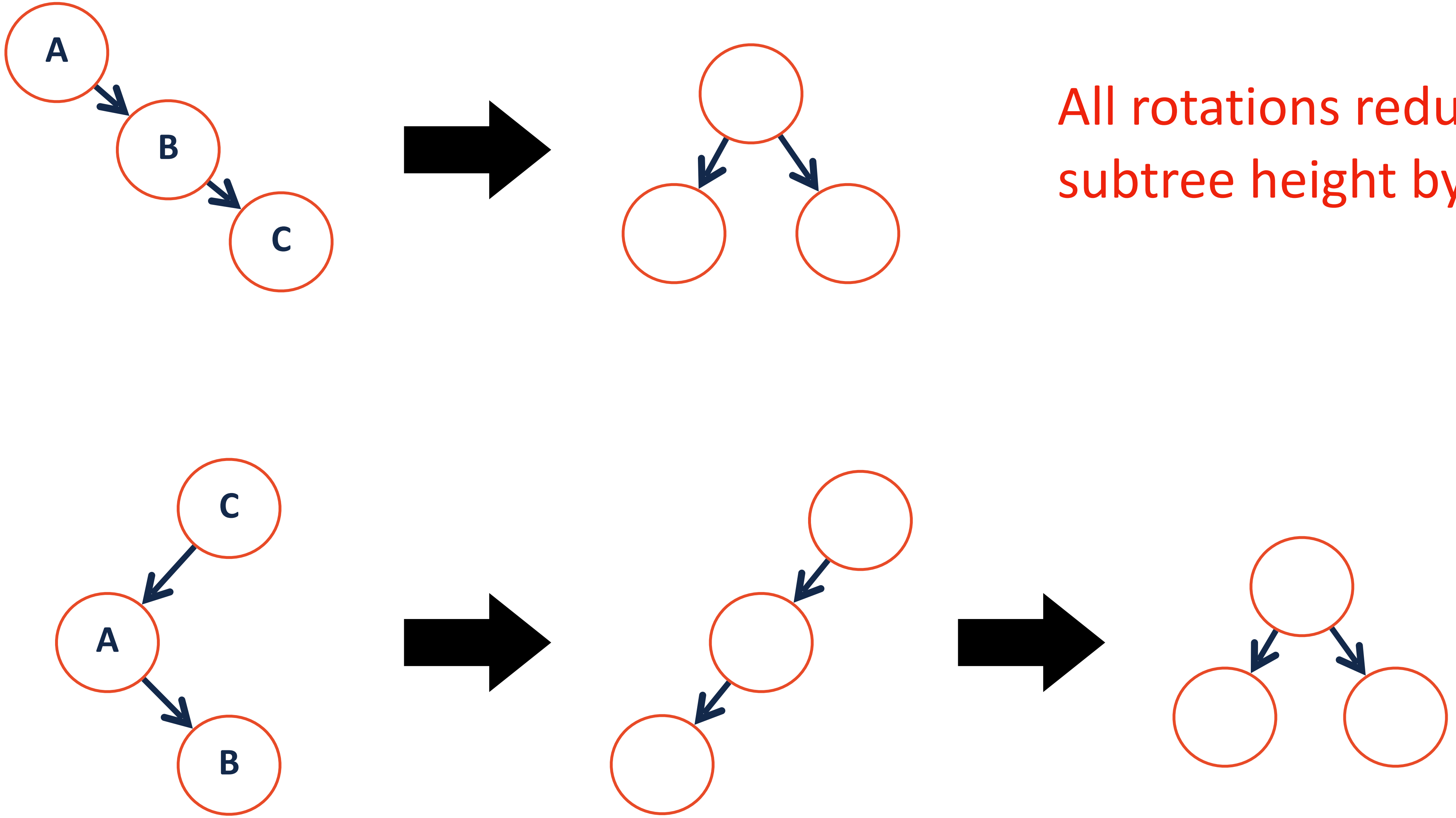


	Left	Right	LeftRight	RightLeft
Root Balance:	2	-2	-2	2
Child Balance:	1	-1	1	-1

AVL Tree Rotations

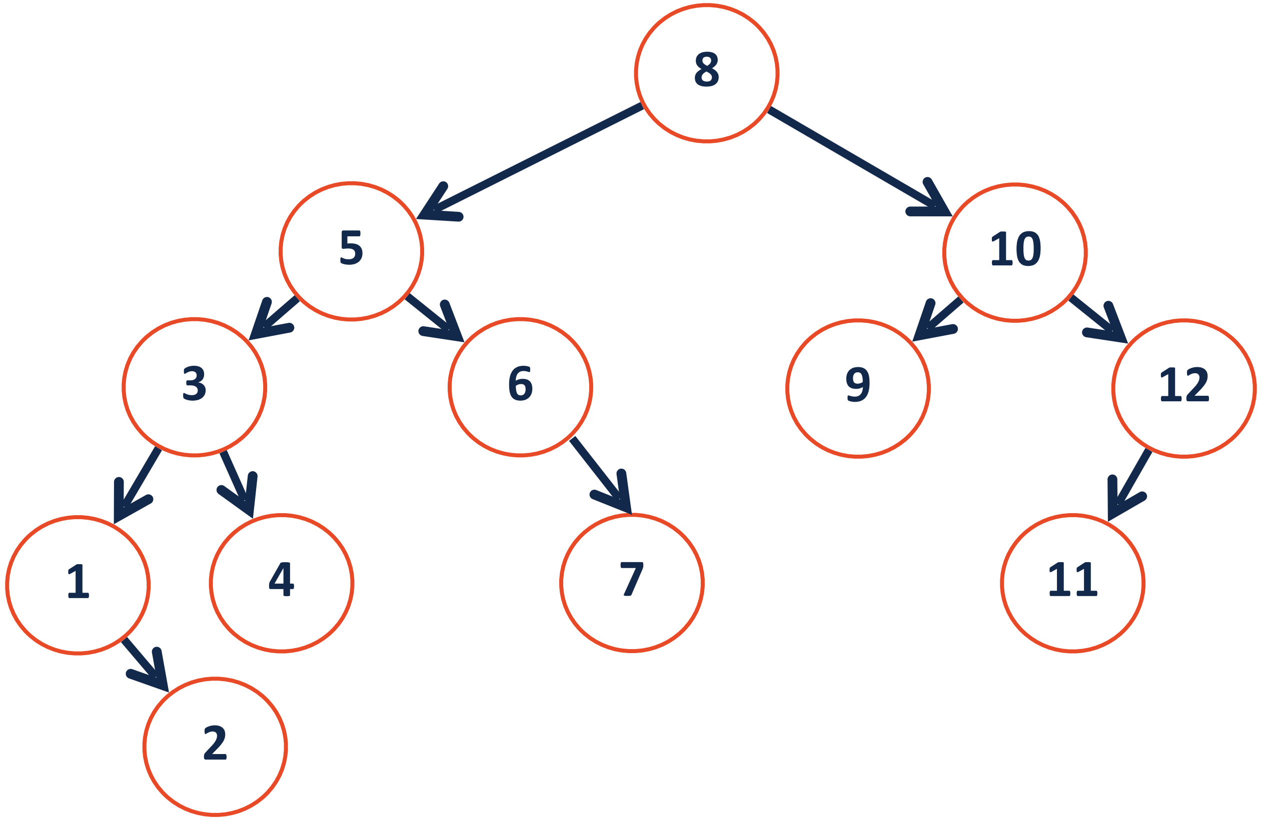
All rotations are O(1)

All rotations reduce subtree height by one



AVL Insertion

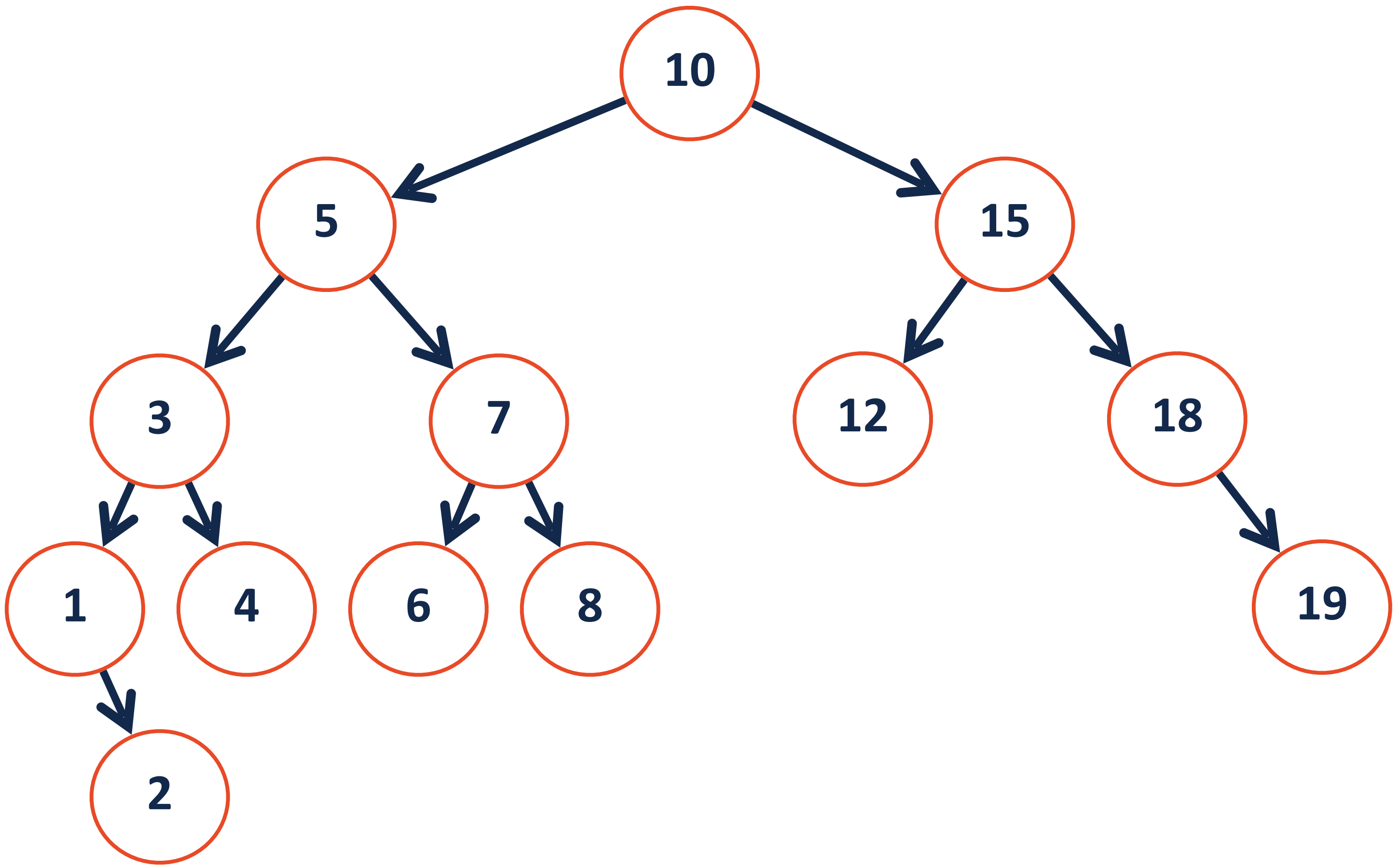
Given an AVL is balanced, insert can insert **at most** one imbalance



AVL Remove

`_remove(12)`

Remove can cause an imbalance at every level



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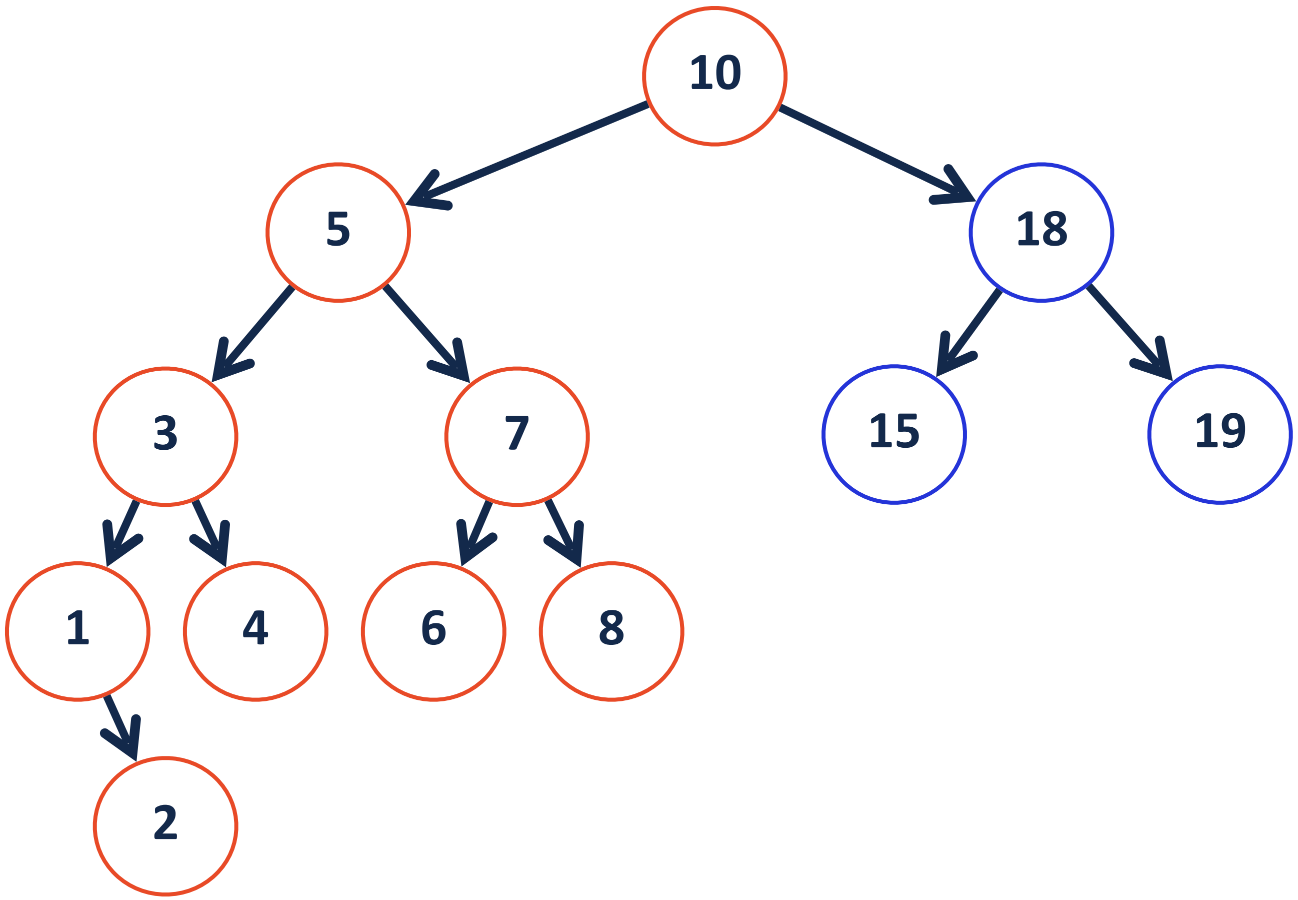

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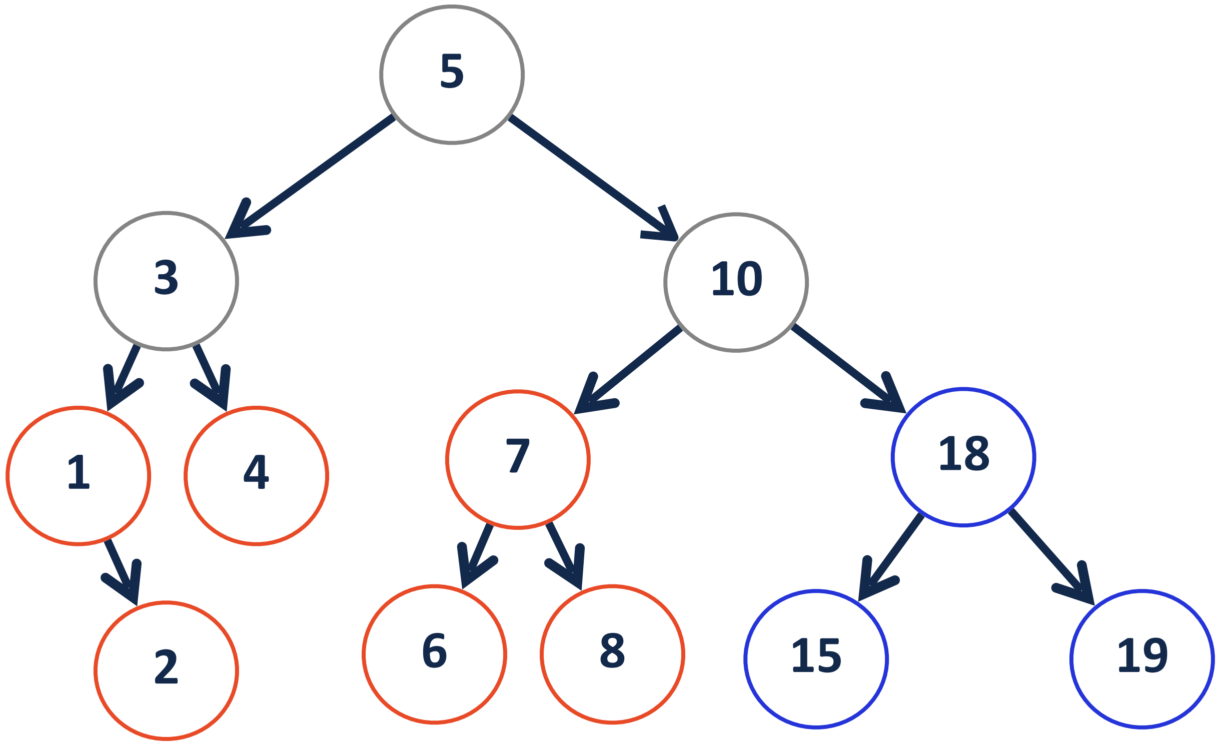

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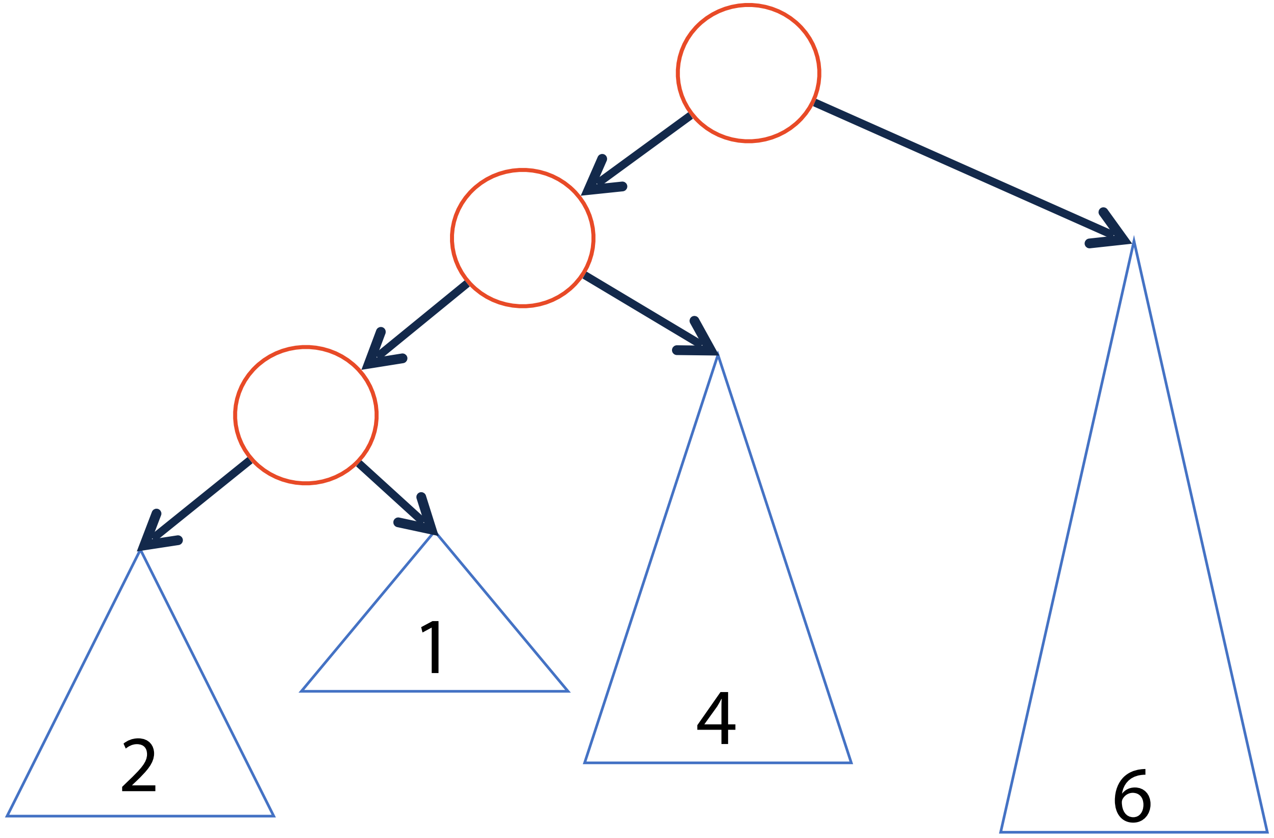

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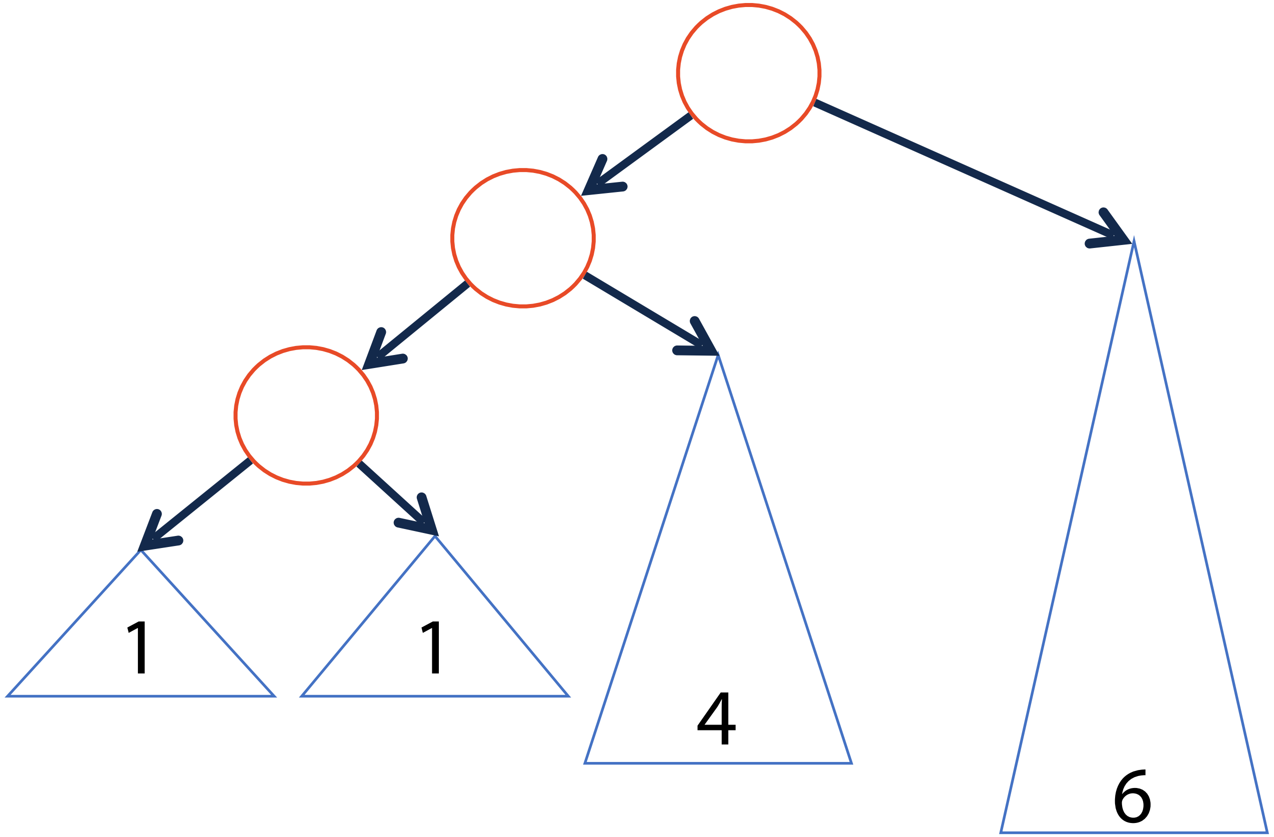

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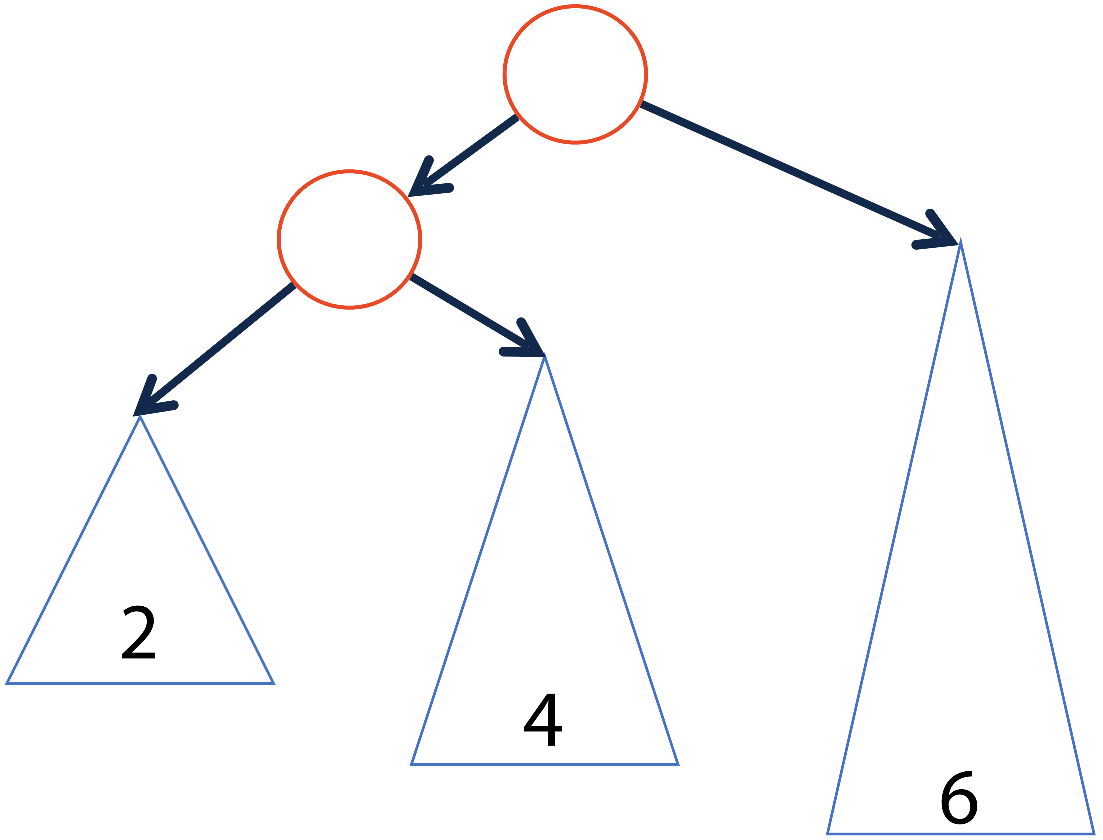

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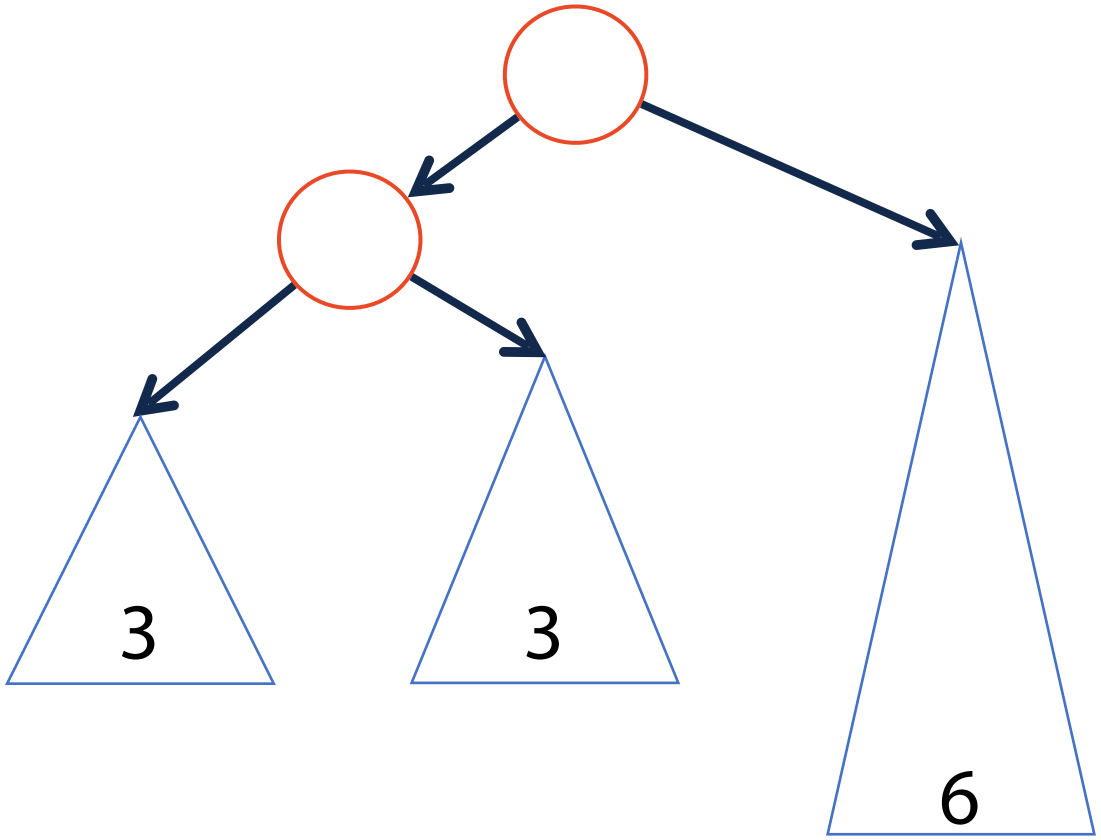

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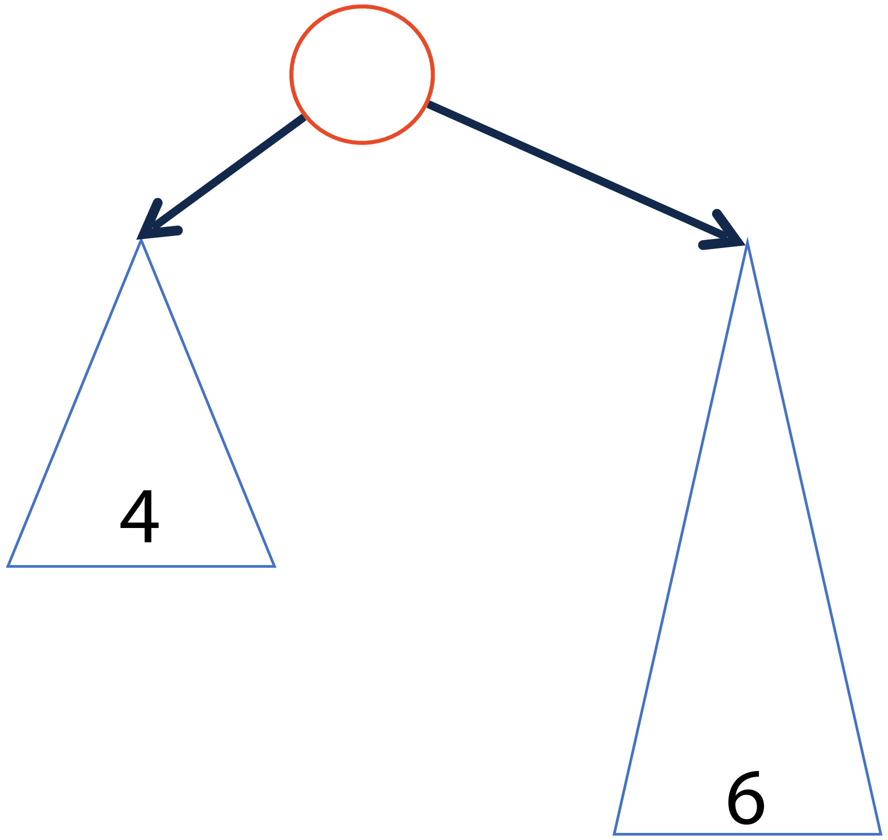

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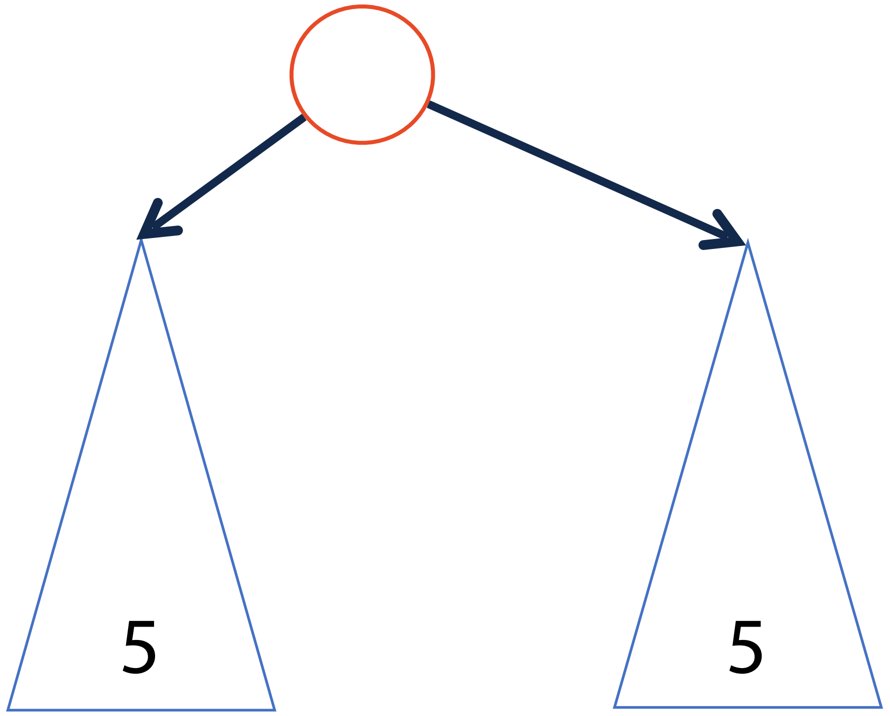


AVL Tree Analysis

For an AVL tree of height h:

Find runs in: ___________.

Insert runs in: ___________.

Remove runs in: ___________.

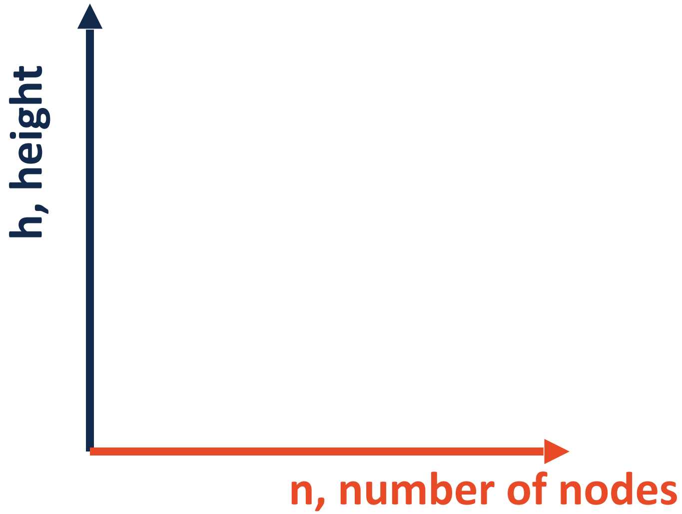
Claim: The height of the AVL tree with n nodes is: **___________.**

AVL Tree Analysis

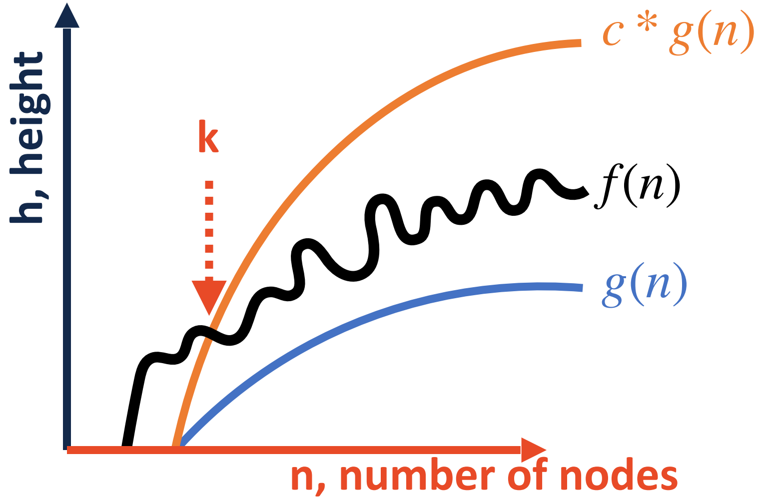
Definition of big-O:

$$f(n) \text{ is } O(g(n)) \text{ iff } \exists c, k \text{ s.t. } f(n) \leq cg(n) \ \forall n > k$$

...or, with pictures:

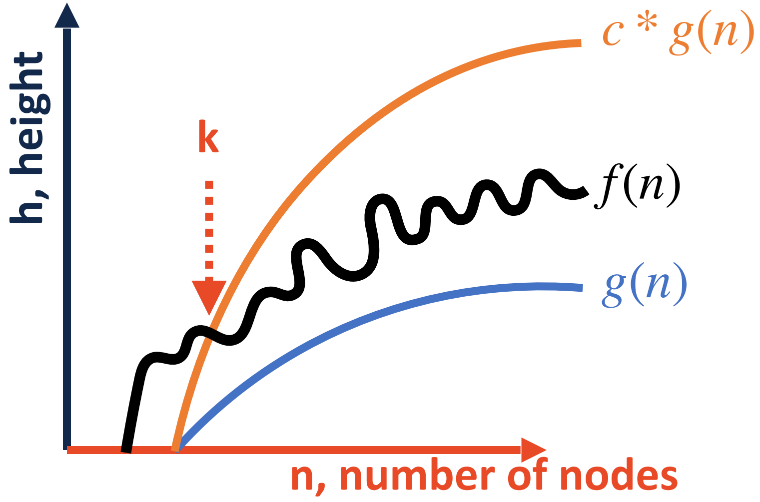


AVL Tree Analysis

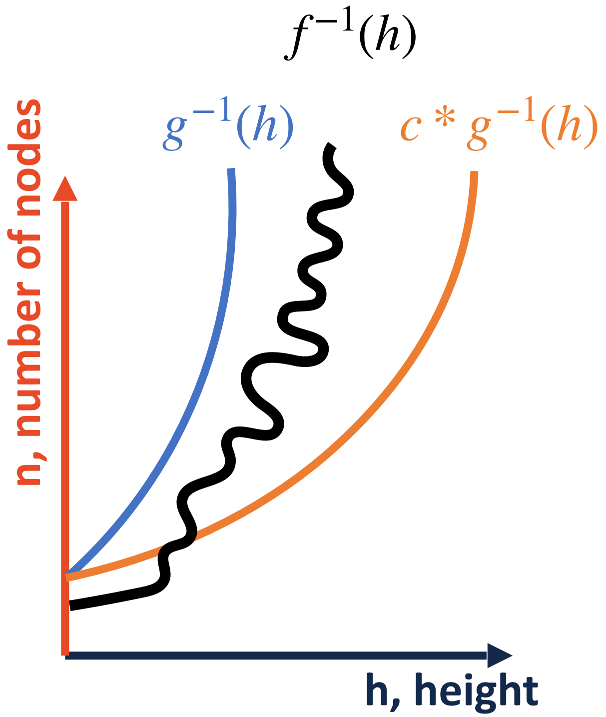


The height of the tree, **f(n)**, will always be less than **c × g(n)** for all values where **n > k**.

AVL Tree Analysis



$f(n)$ = "Tree height given nodes"



$f^{-1}(h)$ = "Nodes in tree given height"

The number of nodes in the tree, **f^{-1}(h)**, will always be <u>greater than</u> **c × g^{-1}(h)** for all values where **n > k**.

Plan of Action

Since our goal is to find the lower bound on **n** given **h**, we can begin by defining a function given **h** which describes the smallest number of nodes in an AVL tree of height **h**:

$N(h)$ = minimum number of nodes in an AVL tree of height h

Simplify the Recurrence

$$N(h) = 1 + N(h-1) + N(h-2)$$

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$$N(h) > N(h-1) + N(h-2)$$

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1) Know characteristic equation? Get answer immediately!

Simplify the Recurrence

$N(h) = 1 + N(h-1) + N(h-2)$

$N(h) > N(h-1) + N(h-2)$

$N(h) > 2N(h-2)$

2) Unroll: $N(h) > 2N(h-2) = 2\left(2(N(h-4)\right) = 2^k\left(N(h-2k)\right)$

Simplify the Recurrence

$N(h) = 1 + N(h-1) + N(h-2)$

$N(h) > N(h-1) + N(h-2)$

$N(h) > 2N(h-2)$

2) Unroll: $N(h) > 2N(h-2) = 2\left(2(N(h-4)\right) = 2^k\left(N(h-2k)\right)$

When $h - 2k = 0, k = h/2$. Thus $N(h) > 2^{h/2}$

Simplify the Recurrence

$$N(h) = 1 + N(h-1) + N(h-2)$$

$$N(h) > N(h-1) + N(h-2)$$

$$N(h) > 2N(h-2)$$

3) Intuit approximate shape from recursion

Simplify the Recurrence

$$N(h) = 1 + N(h-1) + N(h-2)$$

$$N(h) > N(h-1) + N(h-2)$$

$$N(h) > 2N(h-2)$$

By whatever strategy you like: $N(h) > 2^{h/2}$

State a Theorem

Theorem: An AVL tree of height h has at least $N(h) > 2^{h/2}$.

Proof by Induction:

I. Consider an AVL tree and let **h** denote its height.

II. Base Case: _______________

An AVL tree of height ____ has at least ____ nodes.

Prove a Theorem

III. Base Case: _______________

An AVL tree of height ____ has at least ____ nodes.

Prove a Theorem

IV. Induction Step: Assume for all heights $i < h, N(i) \geq 2^{i/2}$.

Prove that $N(h) \geq 2^{h/2}$

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Prove that $N(h) \geq 2^{h/2}$

$$N(h) = 1 + N(h-1) + N(h-2)$$

$$N(h) > 2N(h-2)$$

$$N(h) > 2 * 2^{(h-2)/2}$$

$$N(h) > 2 * 2^{h/2-1}$$

$$N(h) > 2^{h/2}$$

Prove a Theorem

V. Using a proof by induction, we have shown that:

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$N(h) \geq 2^{h/2}$, where $N(h)$ is the **min # of nodes of a tree of height h**

But we need to know n, the **# of nodes in any tree of height h**

Prove a Theorem

V. Using a proof by induction, we have shown that:

$N(h) \geq 2^{h/2}$, where $N(h)$ is the **min # of nodes of a tree of height h**

But we need to know n, the **# of nodes in any tree of height h**

$$n \geq N(h)$$

$$log(n) \geq \frac{h}{2}$$

$$h \leq 2\ log(n)$$

AVL Runtime Proof

An upper-bound on the height of an AVL tree is **O(lg(n))**:

N(h) := Minimum # of nodes in an AVL tree of height h

N(h) = 1 + **N(h-1)** + **N(h-2)**

$> 1 + 2^{(h-1)/2} + 2^{(h-2)/2}$

$> 2 \times 2^{(h-2)/2} = 2^{(h-2)/2+1} = 2^{h/2}$

Theorem #1:

Every AVL tree of height h has at least $2^{h/2}$ nodes.

AVL Runtime Proof

An upper-bound on the height of an AVL tree is **O(lg(n))**:

# of nodes (n) ≥ N(h) > $2^{h/2}$

n > $2^{h/2}$

lg(n) > h/2

2 × lg(n) > h

h < 2 × lg(n) ***, for h ≥ 1***

Proved: The maximum number of nodes in an AVL tree of height h is less than 2 × lg(n).

Summary of Balanced BST

Pros:

Cons:

Every Data Structure So Far

	Unsorted Array	Sorted Array	Unsorted Linked List	Sorted Linked List	Binary Tree	BST	AVL
Find							
Insert							
Remove							
Traverse							