String Algorithms and Data Structures

FM Index

CS 199-225
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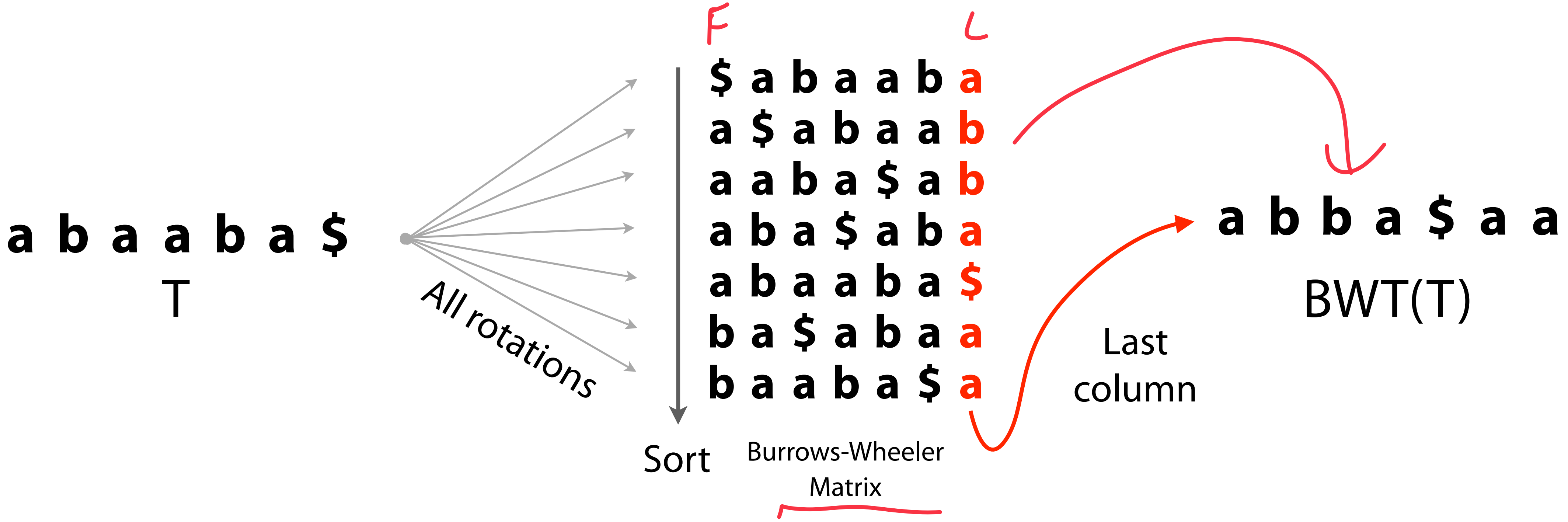
Burrows-Wheeler Transform

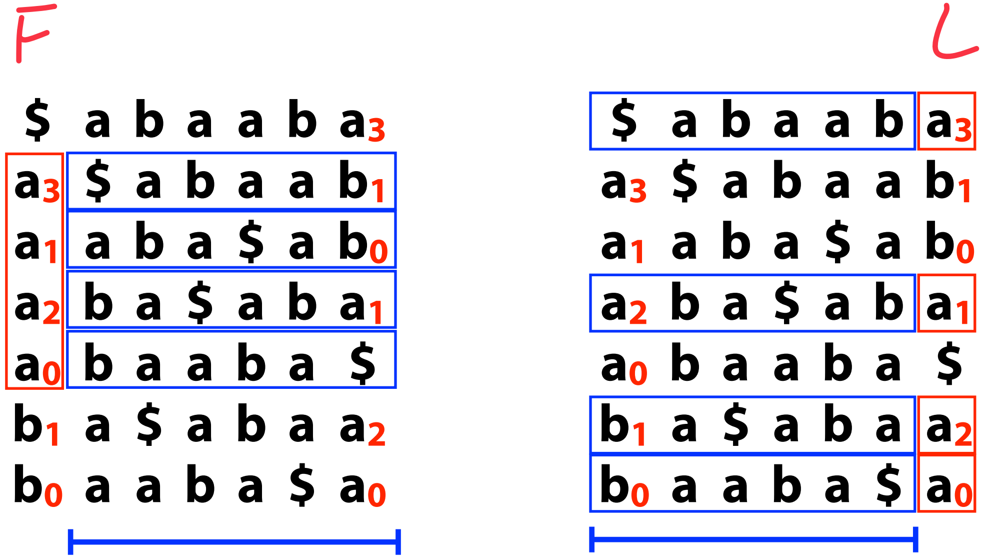
Reversible permutation of the characters of a string



Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform: LF Mapping

The i^{th} occurrence of a character c in L and the i^{th} occurrence of c in F correspond to the *same* occurrence in T (i.e. have same rank)



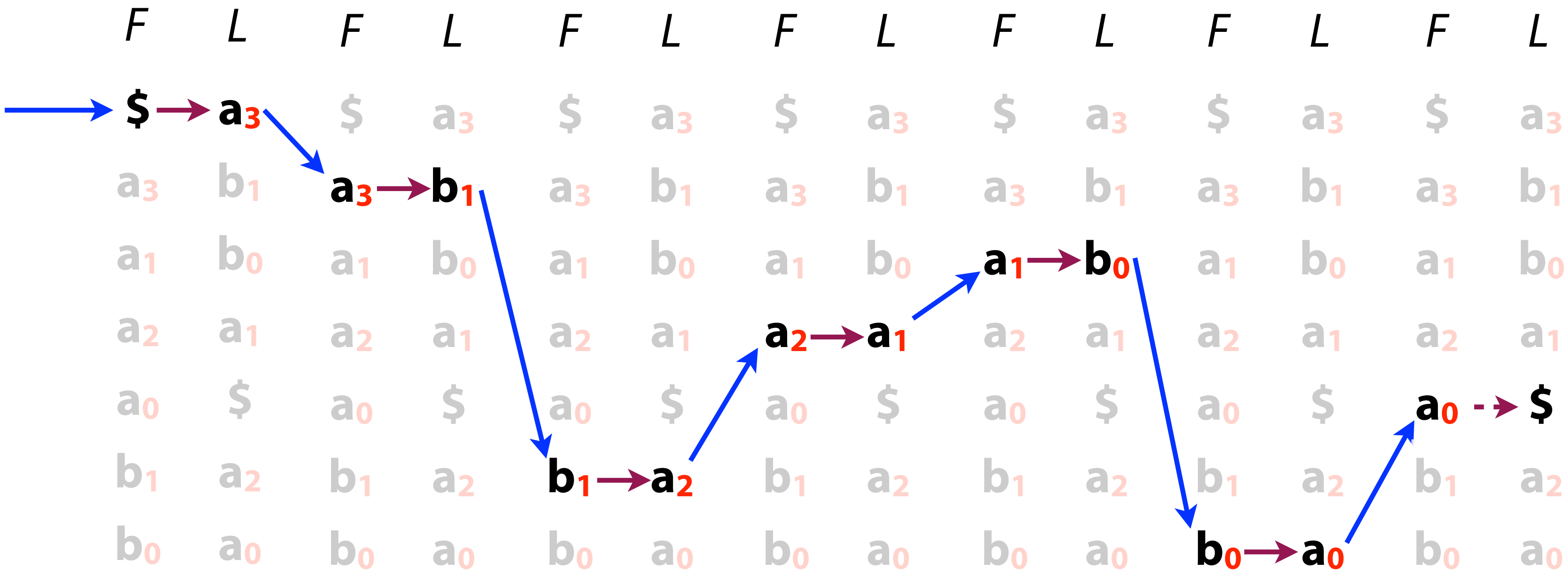
They're sorted by right-context

They're sorted by right-context

Any ranking we give to characters in T will match in F and L

Burrows-Wheeler Transform: LF Mapping

Another way to visualize:



T: a_0 b_0 a_1 a_2 b_1 a_3 $

A review of ‘F' and ‘L’

L = CGGGCC$ Σ = “ACGT”

How can we represent *F*?

↳ Sorted L $ C C C G G G

↳ Run length encoding that is alphabet sorted

A:0, C:3, G:3, T:0

A review of 'F' and 'L'

F is recoverable given L

L = CGGGCC$ Σ = "ACGT"

How can we represent F?

As a full text string: F = $CCCGGG

As a map<string, int>: F = {'$': 1, 'C': 3, 'G': 3}

Implied

A:0, T:0

As a vector<int>: F = [0, 3, 3, 0] *

Both great for search ✓ ✓

O(1)

A review of ‘F' and ‘L’

BWT(T) = `e$lppa`

0 ↓

What row index in *F* contains ‘e’? 2

F : $ a e

What row index in *L* contains ‘e’? 0

L = BWT(T)

What row index in *F* contains the second ‘p’? 5

$ a e l p p

0 1 2 3 4 5

A review of ‘F' and ‘L’

BWT(T) = `e$lppa`

$	a	p	p	l	**e**
a	p	p	l	e	**$**
e	$	a	p	p	**l**
l	e	$	a	p	**p**
p	l	e	$	a	**p**
p	p	l	e	$	**a**

What row index in *F* contains ‘e’? 2

What row index in *L* contains ‘e’? 0

What row index in *F* contains the second ‘p’? 5

FM Index

An index combining the BWT *with a few small auxiliary data structures*

Core of index is ***first (F)*** and ***last (L) rows*** from BWM:

F						*L*
$	a	b	a	a	b	**a**
a	$	a	b	a	a	**b**
a	a	b	a	$	a	**b**
a	b	a	$	a	b	**a**
a	b	a	a	b	a	**$**
b	a	$	a	b	a	**a**
b	a	a	b	a	$	**a**

L is the same size as *T*

F can be represented as array of |Σ| integers (or not stored at all!)

We're discarding *T* — *we can recover it from L!*

O(n) to get rank

FM Index: Querying

F: A:3 B:3

$P =$ **A A A**

By LF mapping before A was A

F - rank
order seen in F

$	B	B	B	A	A	**A_0**
A_0	$	B	B	B	A	**A_1**
A_1	A	$	B	B	B	**A_2**
A_2	A	A	$	B	B	**B_0**
B_0	A	A	A	$	B	**B_1**
B_1	B	A	A	A	$	**B_2**
B_2	B	B	A	A	A	**$**

We know:

1) There is a match
2) the # of matches!..

But no location in T!

FM Index: Querying

$P =$ **B A B**

skip	**\$**	B	B	B	A	A	A_0
\$	A_0	\$	B	B	B	A	A_1
A	A_1	A	\$	B	B	B	A_2
	A_2	A	A	\$	B	B	B_0
4 →	B_0	A	A	A	\$	B	B_1
5	B_1	B	A	A	A	\$	B_2
6	B_2	B	B	A	A	A	\$

As?

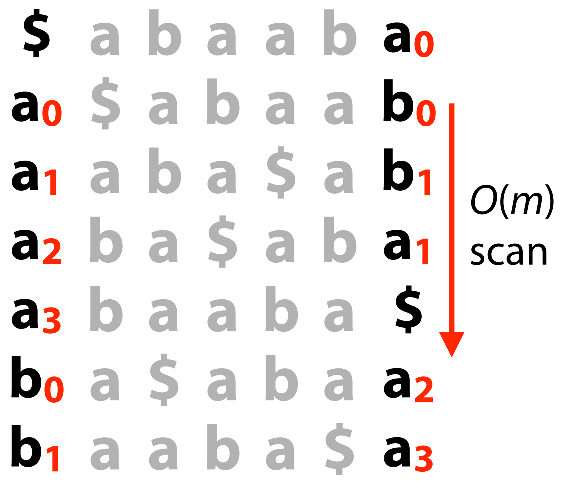
If no matching letter pattern doesnt exist

FM Index: Lingering Issues

1) How are ranks stored? (Fast lookup needed!)

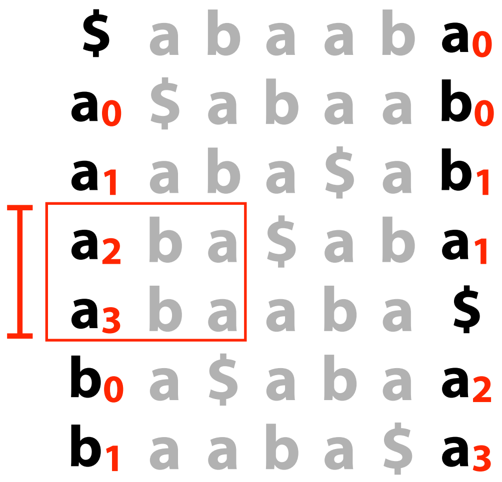
FM Index: Lingering Issues

(1) Scanning for preceding character in *L* is slow



We don't store ranks!

(2) Need way to find where matches occur in *T*:



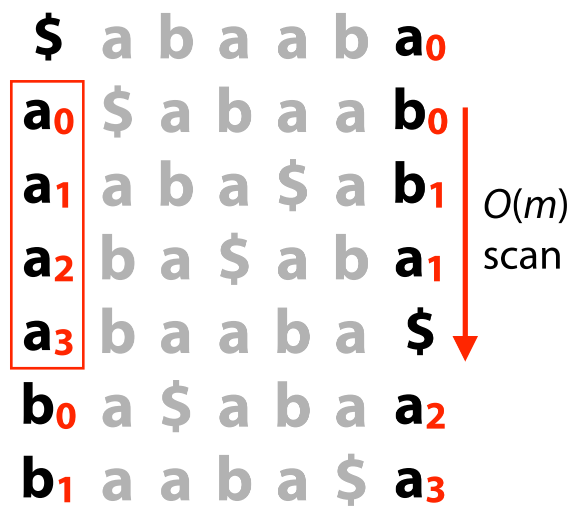
`Current output: [3,4]`

`Location in T: [0,3]`

This is where our auxiliary data structures come in…

FM Index: Fast rank calculations

Is there a fast way to determine which *specific* **b**s precede the **a**s in our range?



More generally, given a range in *L* and a character to search, how can we quickly find all matches (and their ranks)?

FM Index: Occurrence Table

Idea: pre-calculate cumulative # **a**s, **b**s in *L* up to every row:

ε

L	**a**	**b**
a	1	0
b	1	1
b	1	2
a	2	2
$	2	2
a	3	2
a	4	2

FM Index: Occurrence Table

Idea: pre-calculate cumulative # **a**s, **b**s in *L* up to every row:

L	**a**	**b**
a	**1**	0
b	1	**1**
b	1	**2**
a	**2**	2
$	2	2
a	**3**	2
a	**4**	2

FM Index: Occurrence Table

Query: 'aba'

Idea: pre-calculate cumulative # **a**s, **b**s in *L* up to every row:

looking for a

look for b

↳ 2 bs: {b0, b1}

O(1)

F	*L*	**a**	**b**
$	a	1	0
a	b	1	1
a	b	1	2
a	a	2	2
a	$	2	2
b	a	3	2
b	a	4	2

lookup

↳ Gives # character before range

and subtract

↳ Gives # at end of range

FM Index: Occurrence Table

Query: 'aba'

Idea: pre-calculate cumulative # **a**s, **b**s in *L* up to every row:

F	*L*	**a**	**b**	
$	a	1	0	← 0 **b**s up to & including this row
a	b	1	1	
a	b	1	2	
a	a	2	2	
a	$	2	2	← 2 **b**s up to & including this row
b	a	3	2	
b	a	4	2	

FM Index: Occurrence Table

Query: 'aba'

Idea: pre-calculate cumulative # **a**s, **b**s in *L* up to every row:

F	*L*	**a**	**b**
$	**a**	**1**	**0**
a	**b**	**1**	**1**
a	**b**	**1**	**2**
a	**a**	**2**	**2**
a	**$**	**2**	**2**
b	**a**	**3**	**2**
b	**a**	**4**	**2**

What values of **a** (including rank) should I look up next?

a_2, a_3

Seen a_0, a_1

2 as in range

O(1) time

FM Index: Occurrence Table

Query: 'bb'

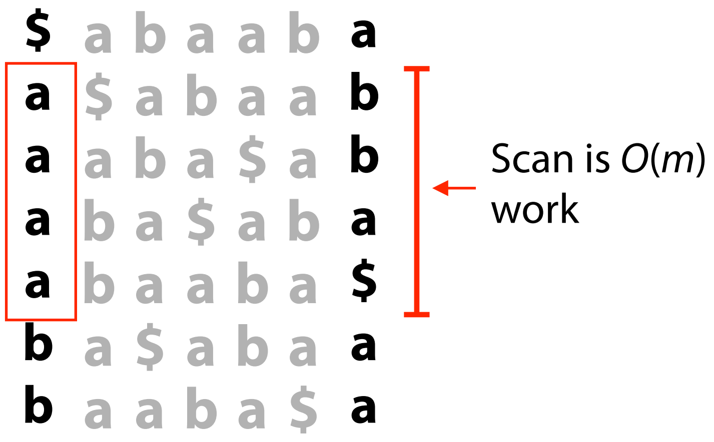
What two indices should I look up? What ranks did we find?

F	*L*	a	b
$	a	1	0
a	b	1	1
a	$	1	1
b	b	1	2
b	b	1	3
b	b	1	4
b	a	2	4

FM Index: Occurrence Table

An index combining the BWT *with a few small auxiliary data structures*

Occurrence table speeds up L lookup by implicitly storing **ranks**



a	b
1	0
1	1
1	2
2	2
2	2
3	2
4	2

Rows: m; columns: $|\Sigma|$

Lookup is O(1) work

Table is $m \times |\Sigma|$ integers — that's worse than a suffix array!

FM Index: Occurrence Table

Next idea: pre-calculate # **a**s, **b**s in *L* up to *some* rows, e.g. every 5th row.
Call pre-calculated rows *checkpoints*.

X

Some constant

F	*L*	**a**	**b**
$	**a**	**1**	**0**
a	**b**		
a	**b**		
a	**a**		
a	**$**		
b	**a**	**3**	**2**
b	**a**		

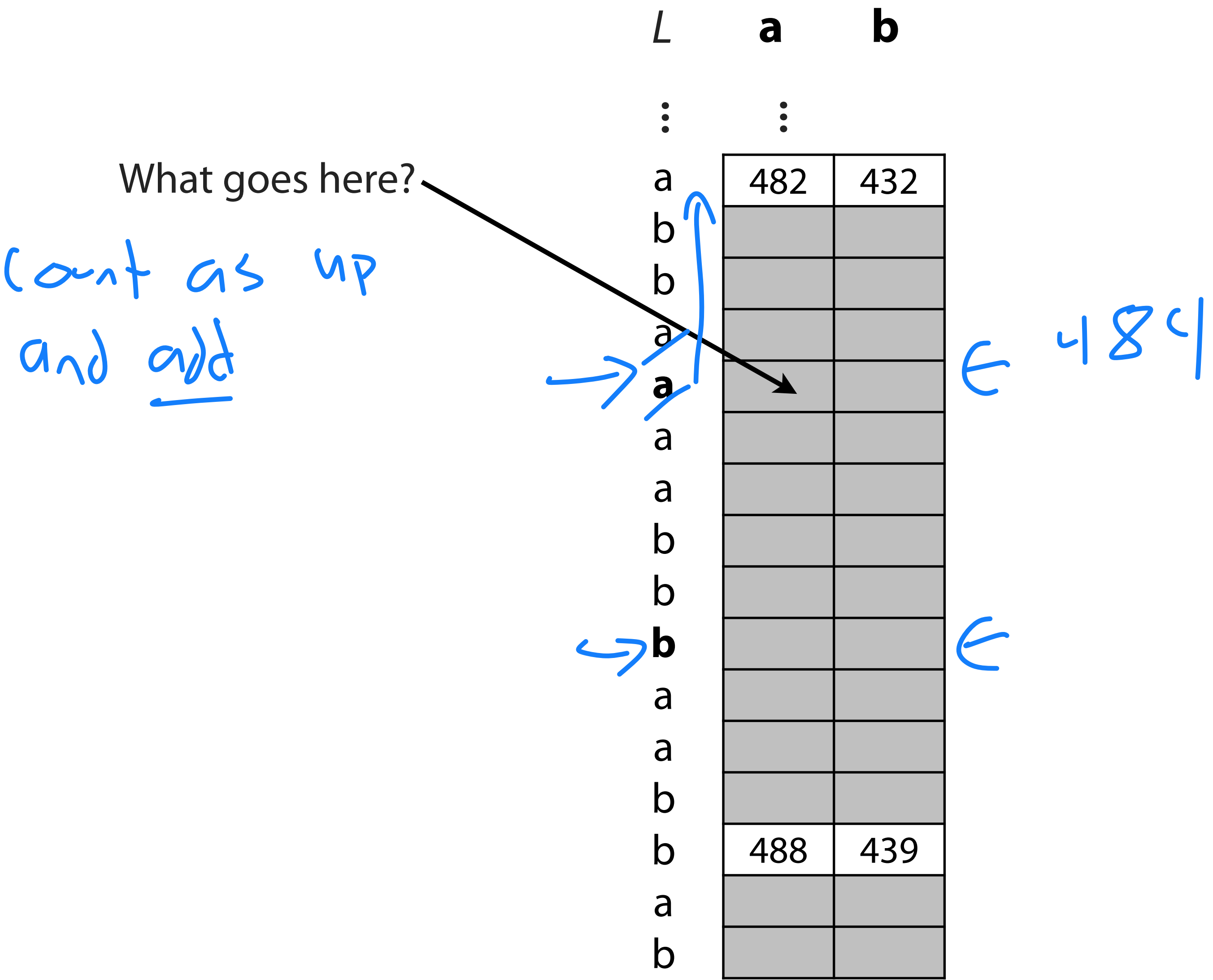
FM Index: Occurrence Table

To resolve a lookup for a non-checkpoint row, walk to nearest checkpoint. Use value at that checkpoint, *adjusted for characters we saw along the way*.

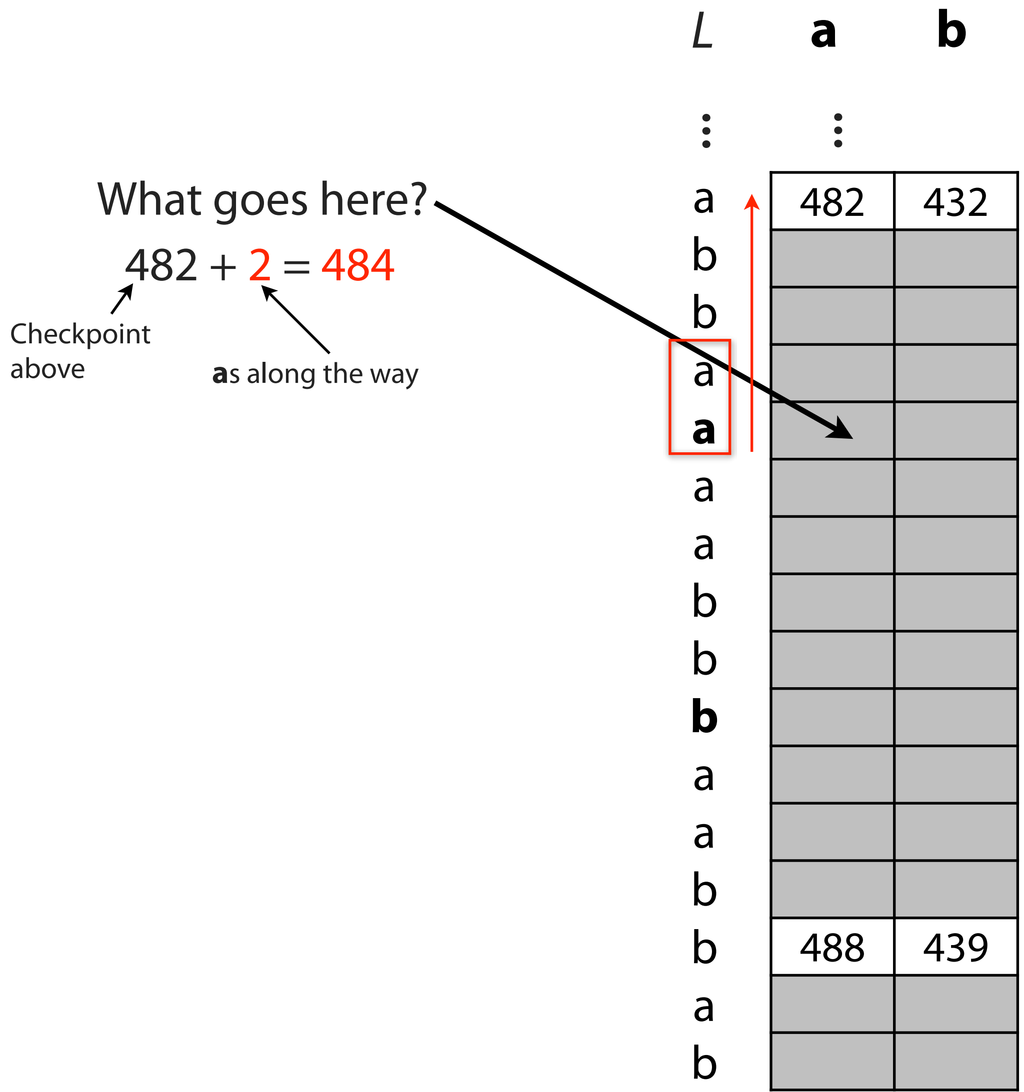
F	*L*	**a**	**b**
$	**a**	**1**	**0**
a	**b**		
a	**b**		
a	**a**		
a	**$**		
b	**a**	**3**	**2**
b	**a**		

walk to nearest checkpoint

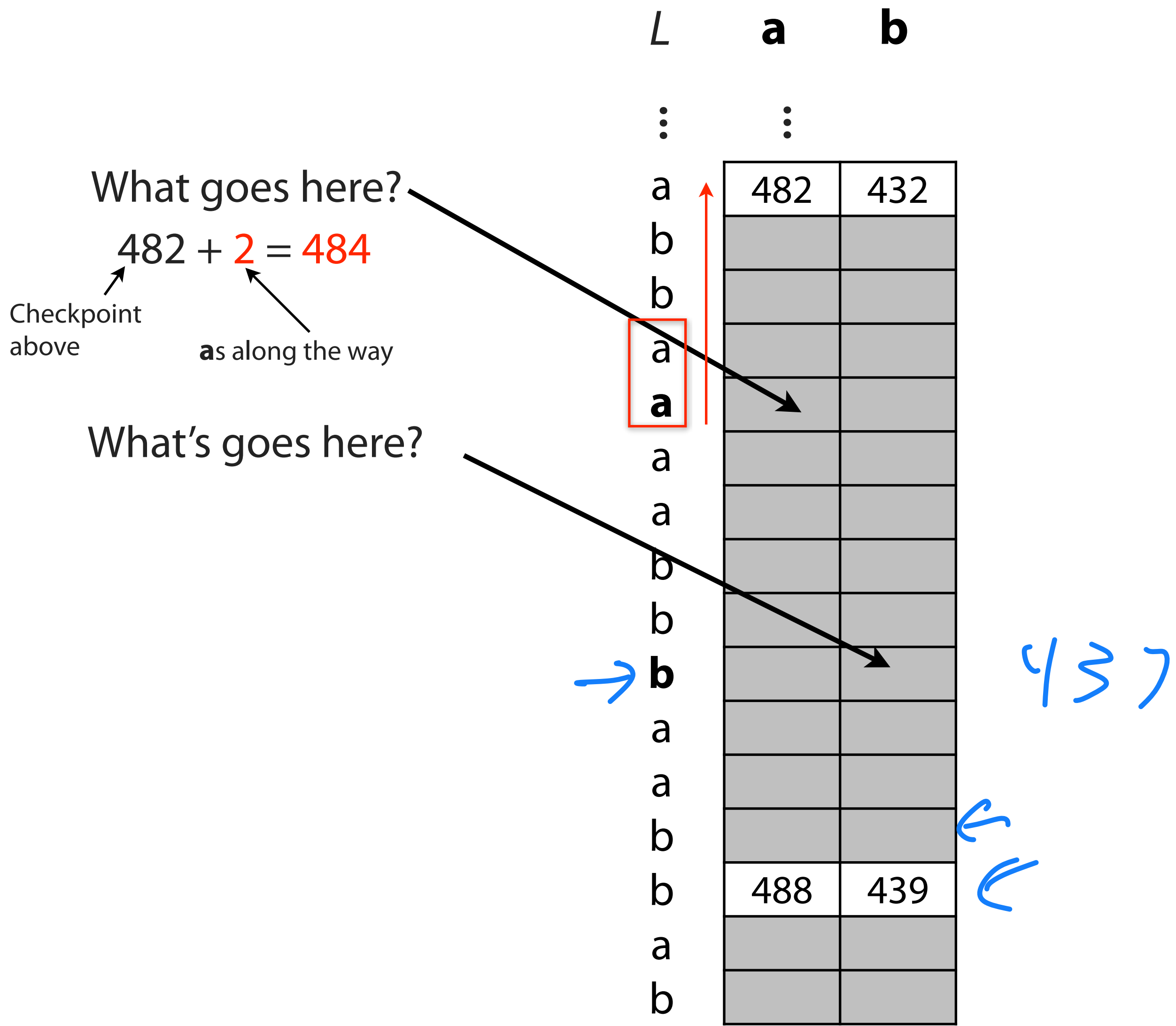
FM Index: Occurrence Table



FM Index: Occurrence Table



FM Index: Occurrence Table



FM Index: Occurrence Table

What goes here?

$482 + 2 = 484$

Checkpoint above

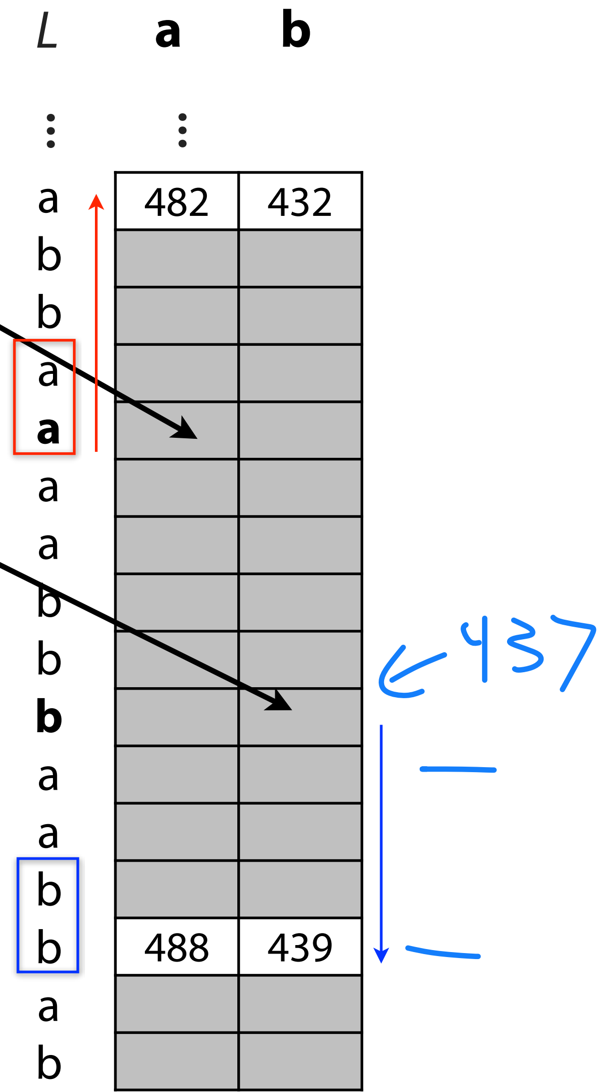
as along the way

What's goes here?

$439 - 2 = 437$

Checkpoint below

bs along the way

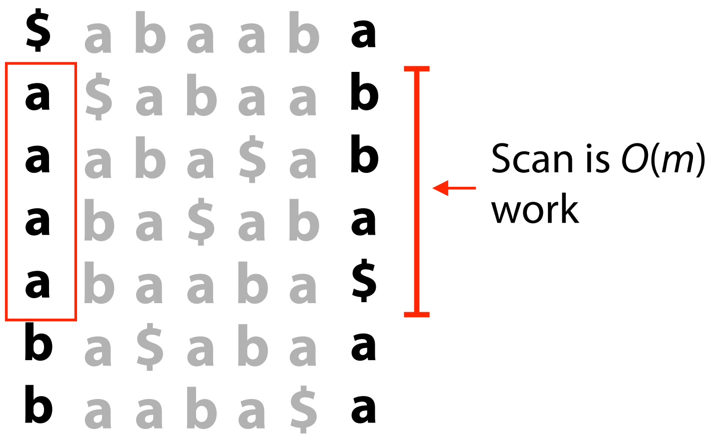


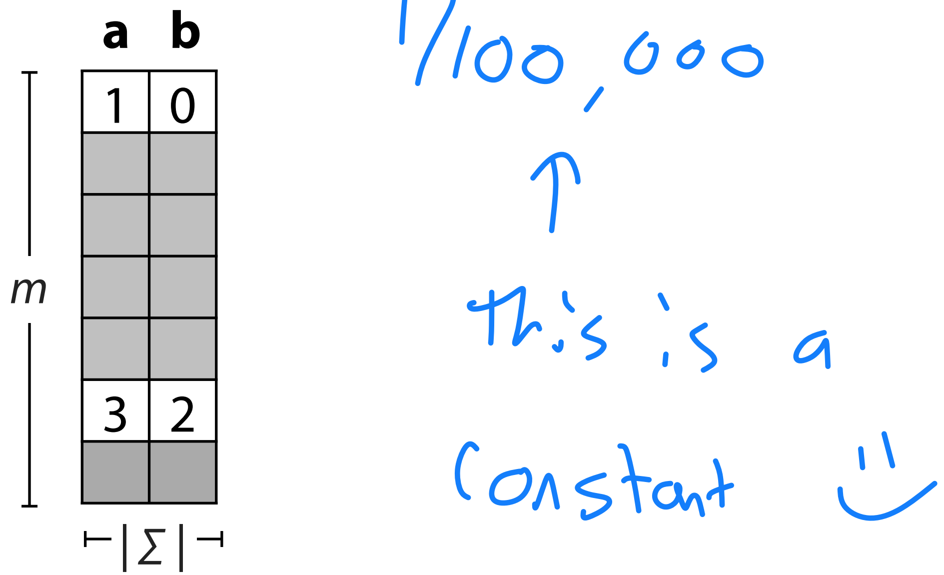
If checkpoints are $O(1)$ distance apart, lookups are $O(1)$

FM Index: Occurrence Table

An index combining the BWT *with a few small auxiliary data structures*

Occurrence table speeds up *L* lookup by implicitly storing **ranks**





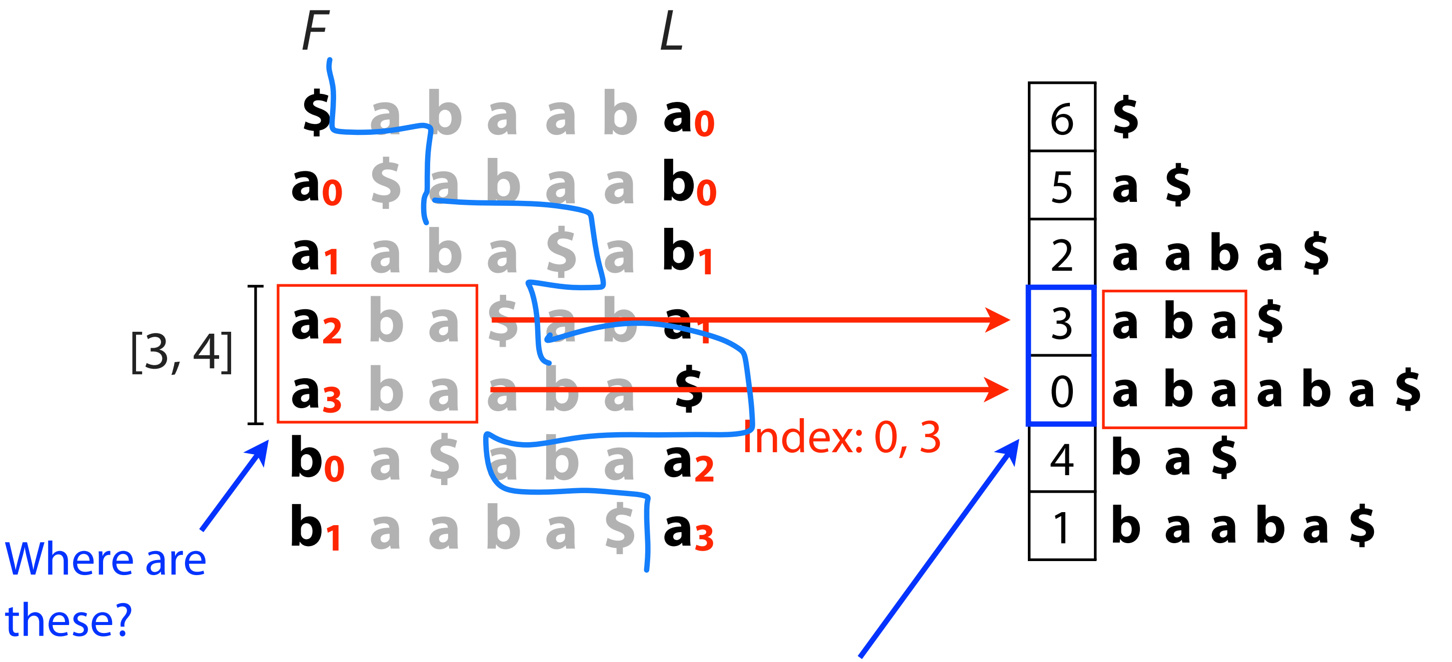
Checkpoints reduce the storage costs (Still O(m) but better than SA)

FM Index: Querying

Problem 2: We don't know *where* the matches are in T...

P = **aba**

Got the same range, [3, 4], we would have got from suffix array



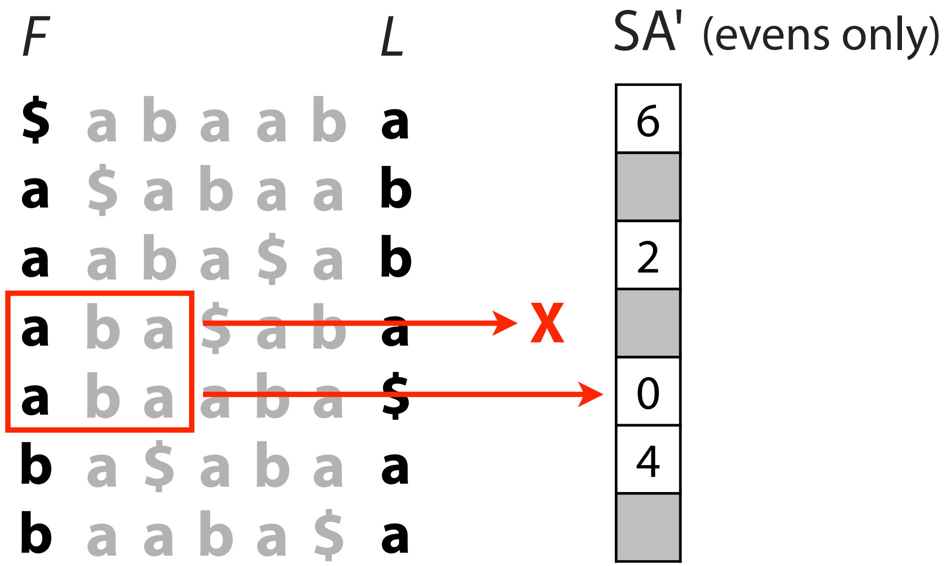
FM Index: Suffix Array Sampling

Idea: store some suffix array elements, but not all

F		*L*	SA' (evens only)
$	a b a a b	**a**	6
a	$ a b a a	**b**	
a	a b a $ a	**b**	2
a	b a $ a b	**a**	
a	b a a b a	**$**	0
b	a $ a b a	**a**	4
b	a a b a $	**a**	

FM Index: Suffix Array Sampling

Idea: store some suffix array elements, but not all

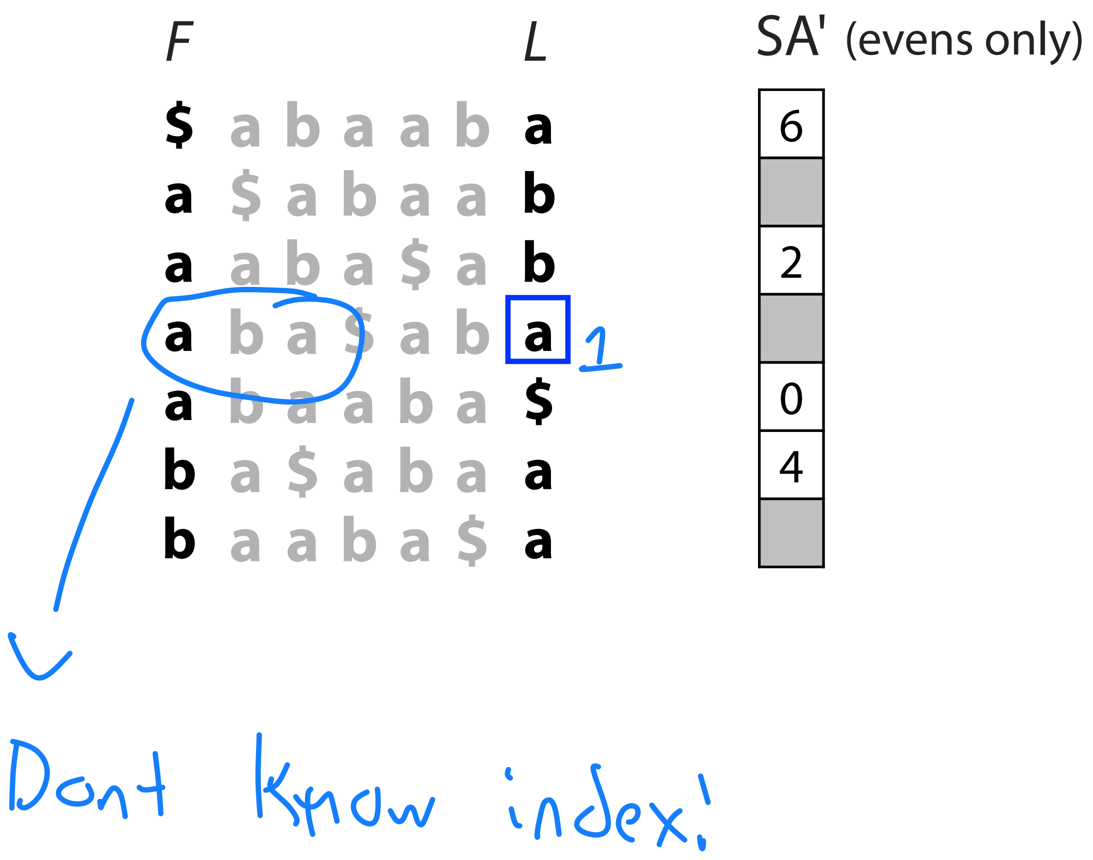


Lookup for row 4 succeeds

Lookup for row 3 fails - SA entry was discarded

FM Index: Suffix Array Sampling

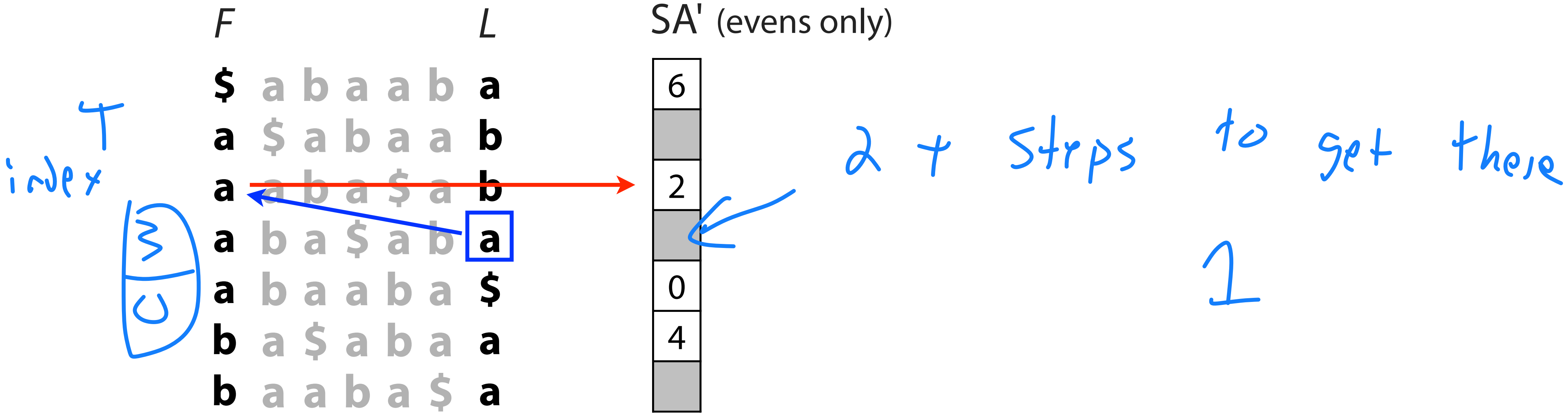
LF Mapping tells us that "**a**" at the end of row 3 corresponds to...



FM Index: Suffix Array Sampling

LF Mapping tells us that "**a**" at the end of row 3 corresponds to...

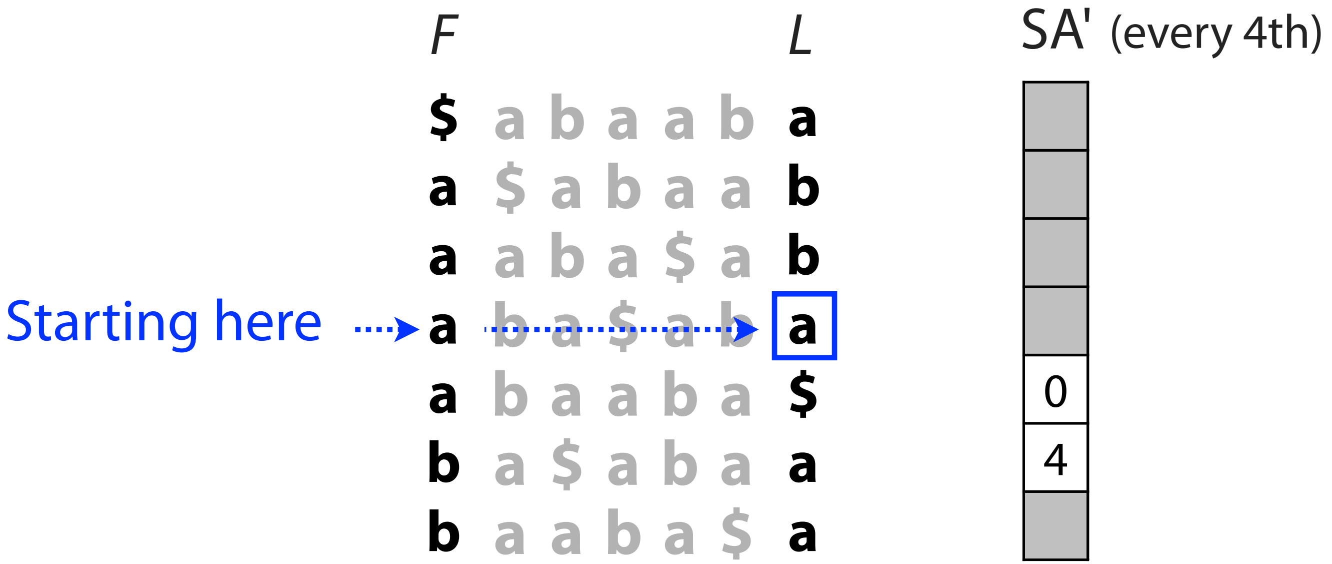
... "**a**" at the beginning of row 2



If saved SA values are O(1) positions apart in *T*, resolving index is O(1) time

FM Index: Suffix Array Sampling

Many LF-mapping steps may be required to get to a sampled row:



FM Index: Suffix Array Sampling

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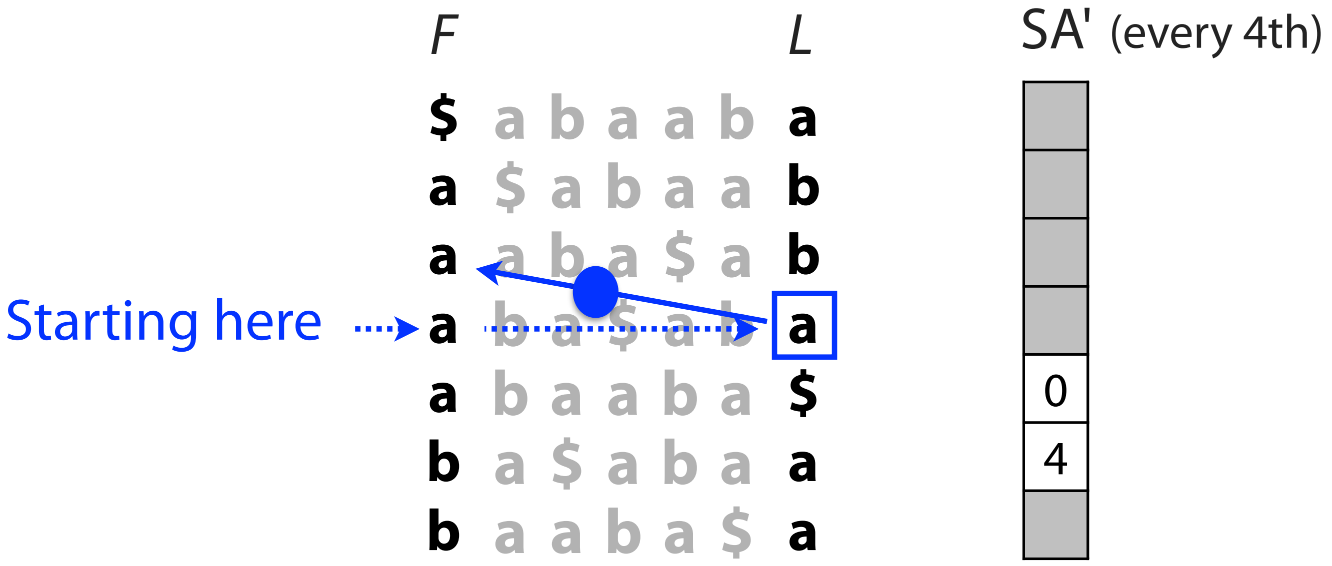

FM Index: Suffix Array Sampling

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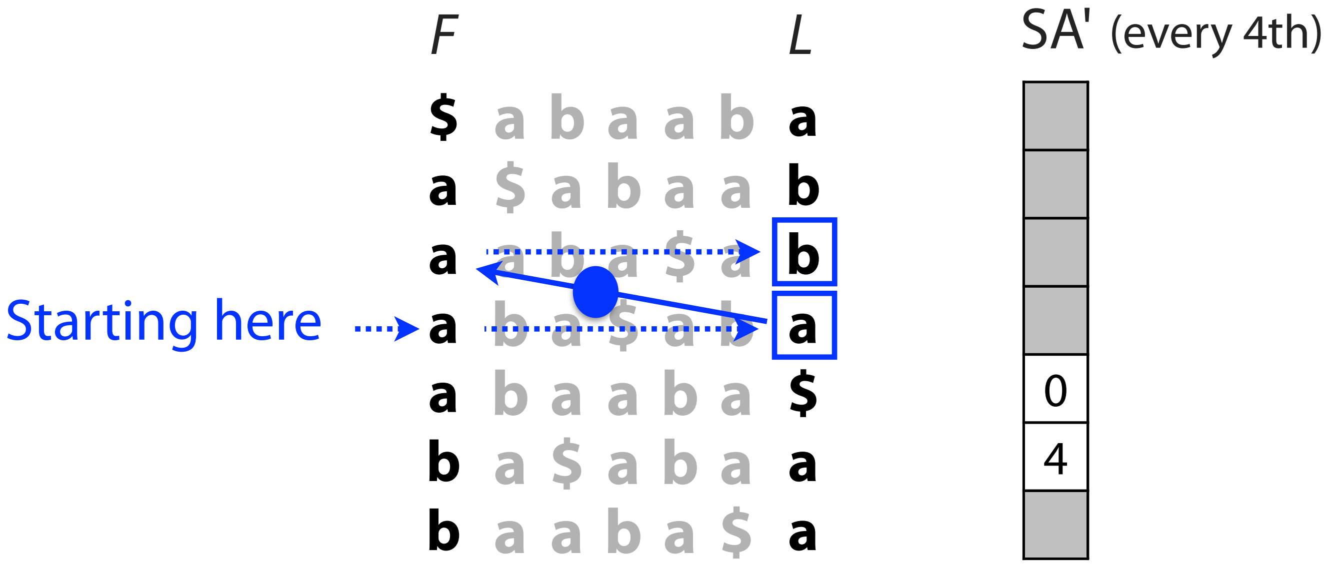

FM Index: Suffix Array Sampling

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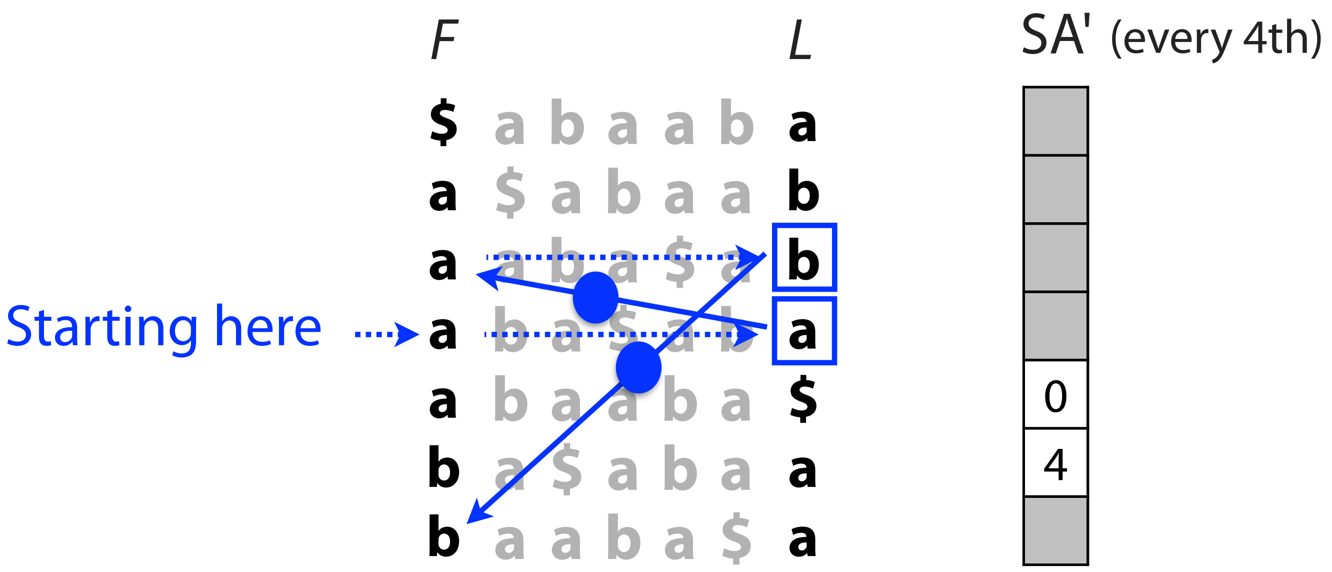
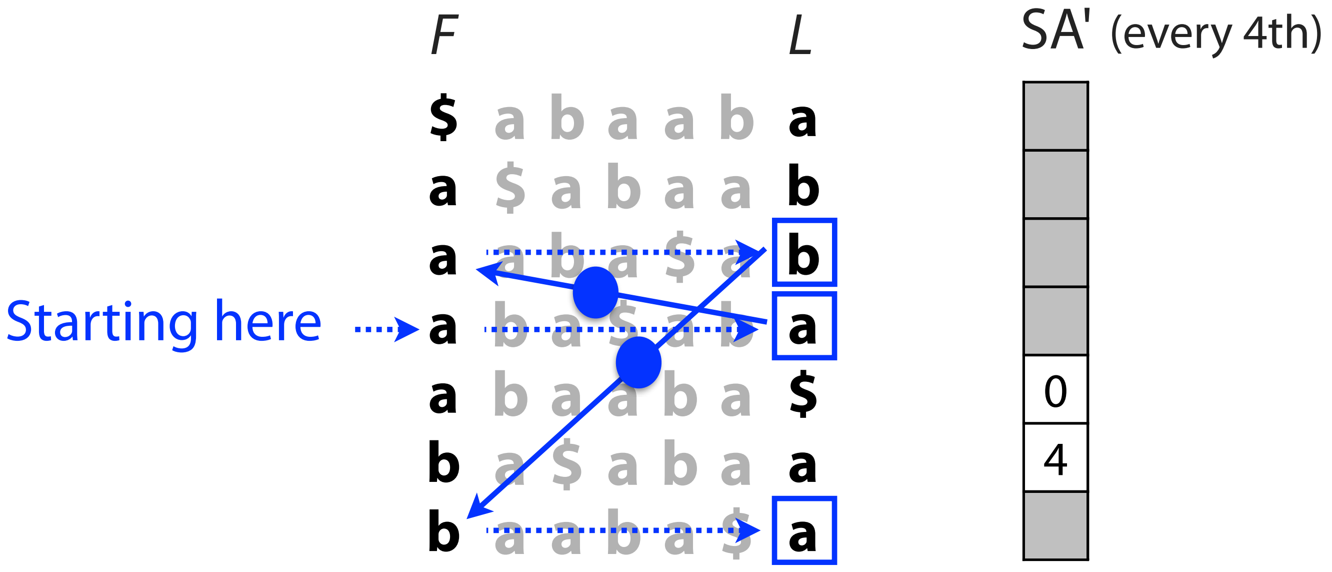


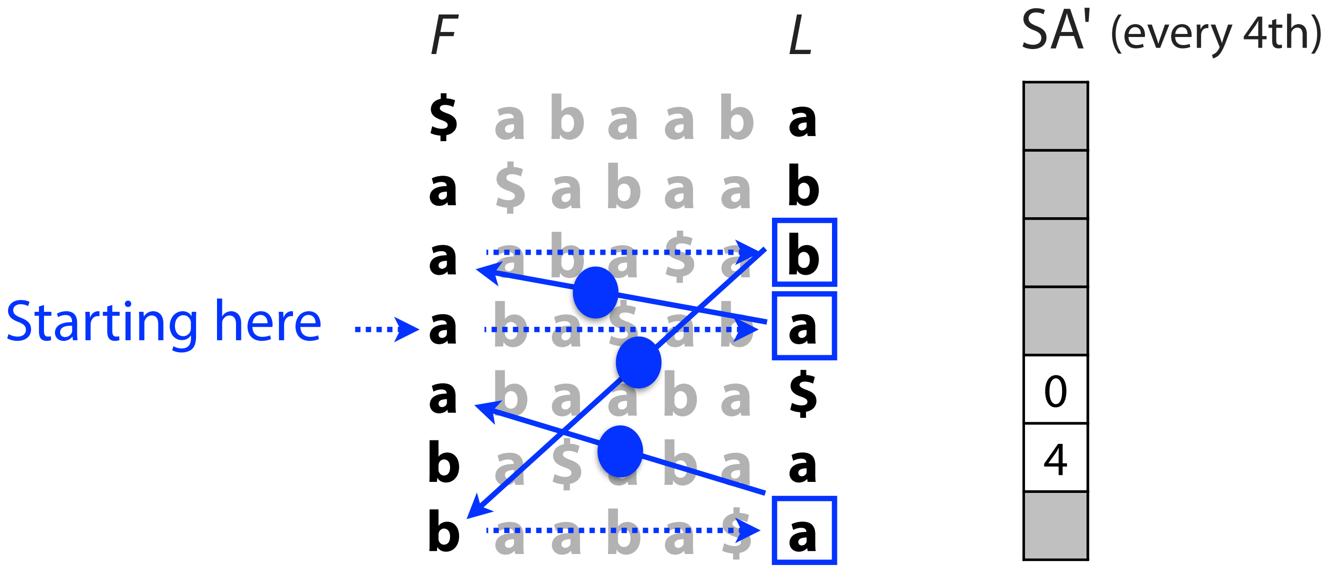
FM Index: Suffix Array Sampling

Many LF-mapping steps may be required to get to a sampled row:



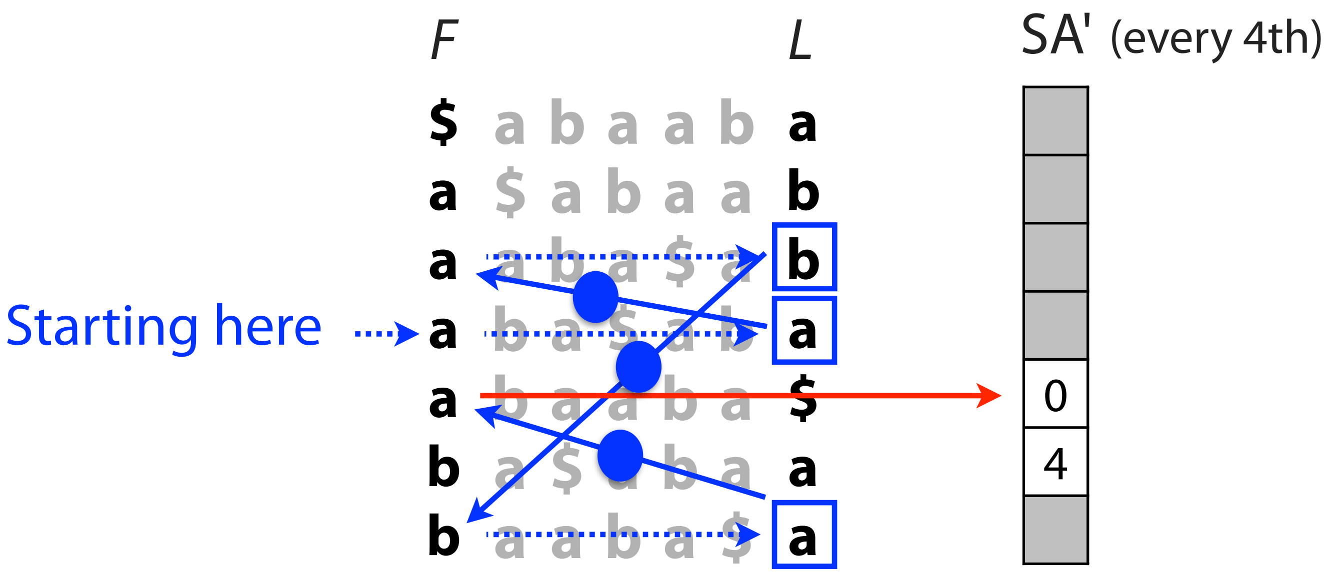
FM Index: Suffix Array Sampling

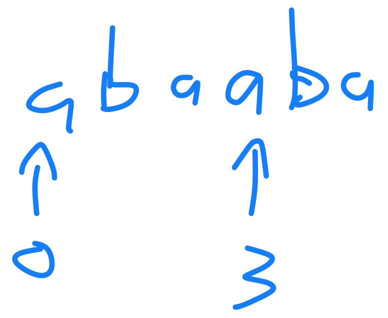
Many LF-mapping steps may be required to get to a sampled row:



FM Index: Suffix Array Sampling

Many LF-mapping steps may be required to get to a sampled row:



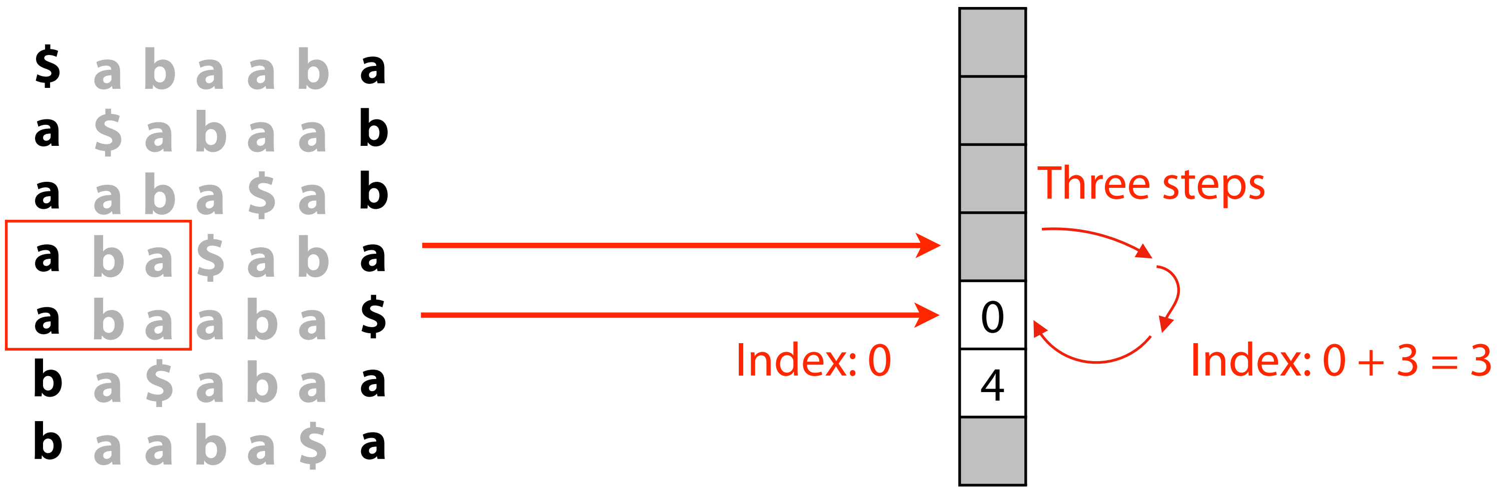


Missing value = 0 (SA val at destination) + 3 (# steps to destination) = **3**

FM Index: Suffix Array Sampling

An index combining the BWT *with a few small auxiliary data structures*

Stores all index positions in T with O(1) extra work to calculate



Lets put all these pieces together…

FM Index: Querying

P = **aba**

get_frange()

F						*L*
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b
a_1	a	b	a	$	a	b
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b	a	$	a	b	a	a_2
b	a	a	b	a	$	a_3

```
pair<int, int> get_frange(string c, int s, int e)
```

Input:

- `string c` : The char we are looking for in *F*
- `int s` : The starting ***rank*** value
- `int e` : The ending ***rank*** value

Output:

- A pair of values (index start, index end)

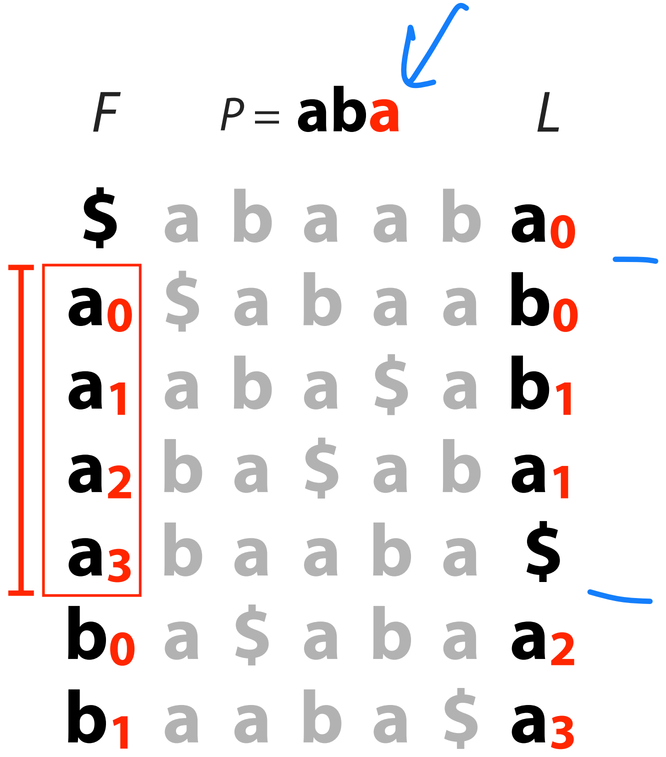
What are c, s, and e?

c = a, s = 0, e = 3

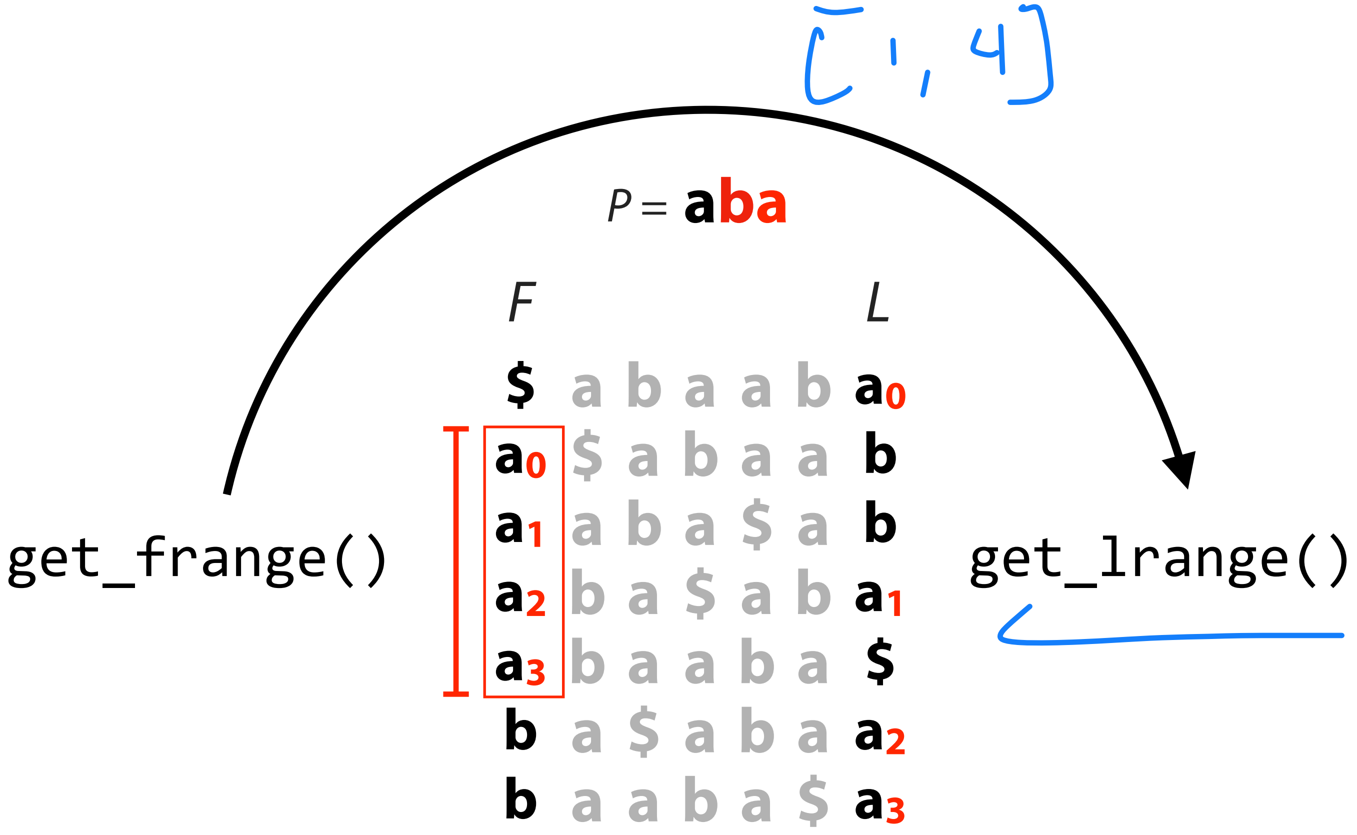
#s from F

What are the output values?

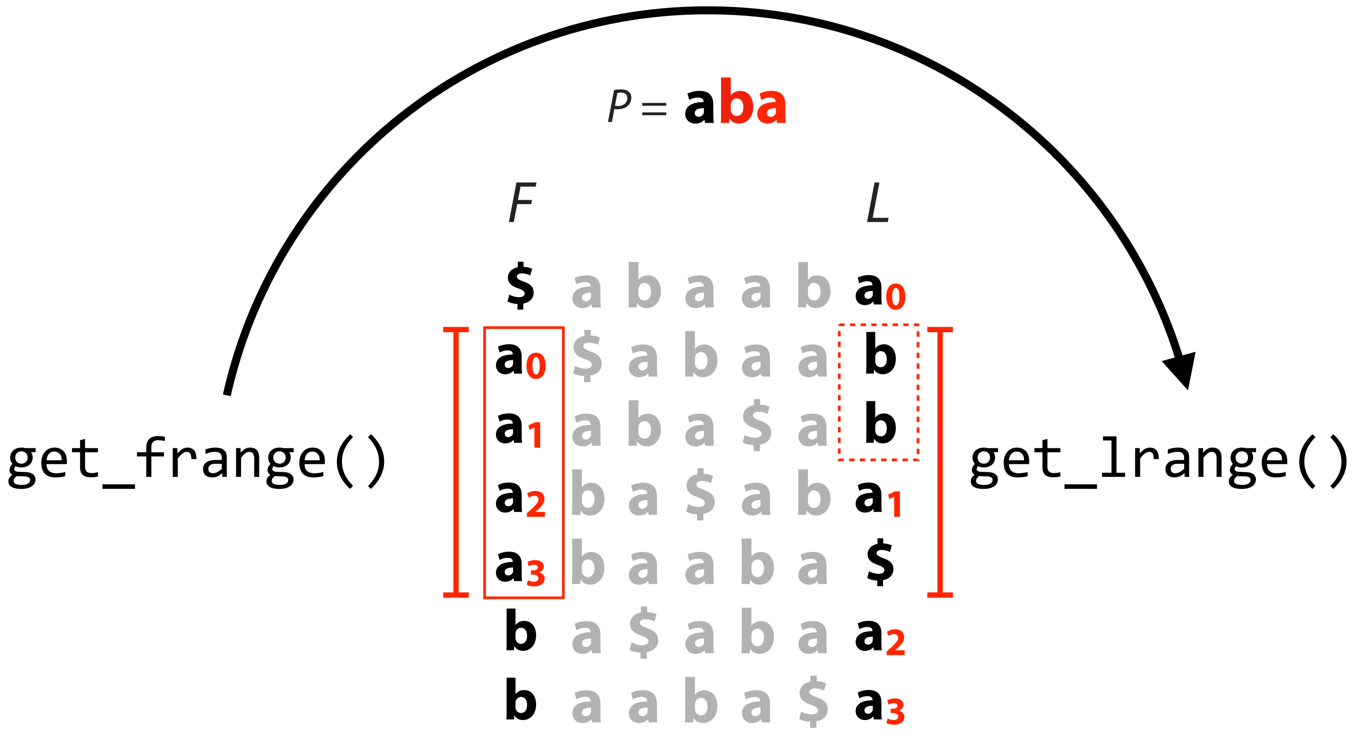
[1, 4]



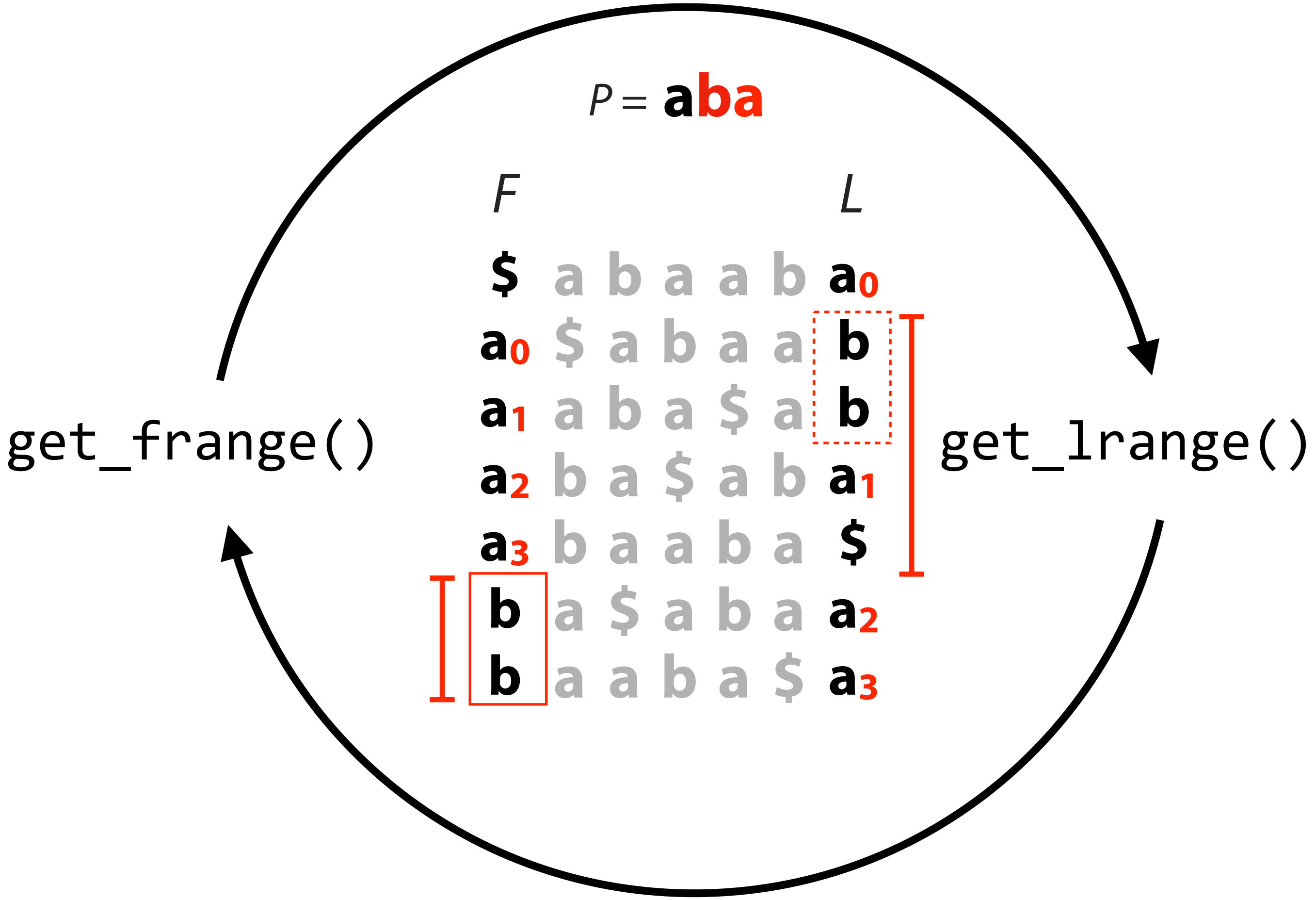
FM Index: Querying



FM Index: Querying



FM Index: Querying



```
pair<int, int> get_lrange(string c, int s, int e)
```

Input:

- **`string c`**: The char we are looking for in *F*
- **`int s`**: The starting ***index*** of our range
- **`int e`**: The ending ***index*** of our range

Output:

- A pair of values (# occurrences start, end)

P = **aba**

F						L
\$	a	b	a	a	b	a_0
a_0	\$	a	b	a	a	b_0
a_1	a	b	a	\$	a	b_1
a_2	b	a	\$	a	b	a_1
a_3	b	a	a	b	a	\$
b_0	a	\$	a	b	a	a_2
b_1	a	a	b	a	\$	a_3

Before 1

at end 4

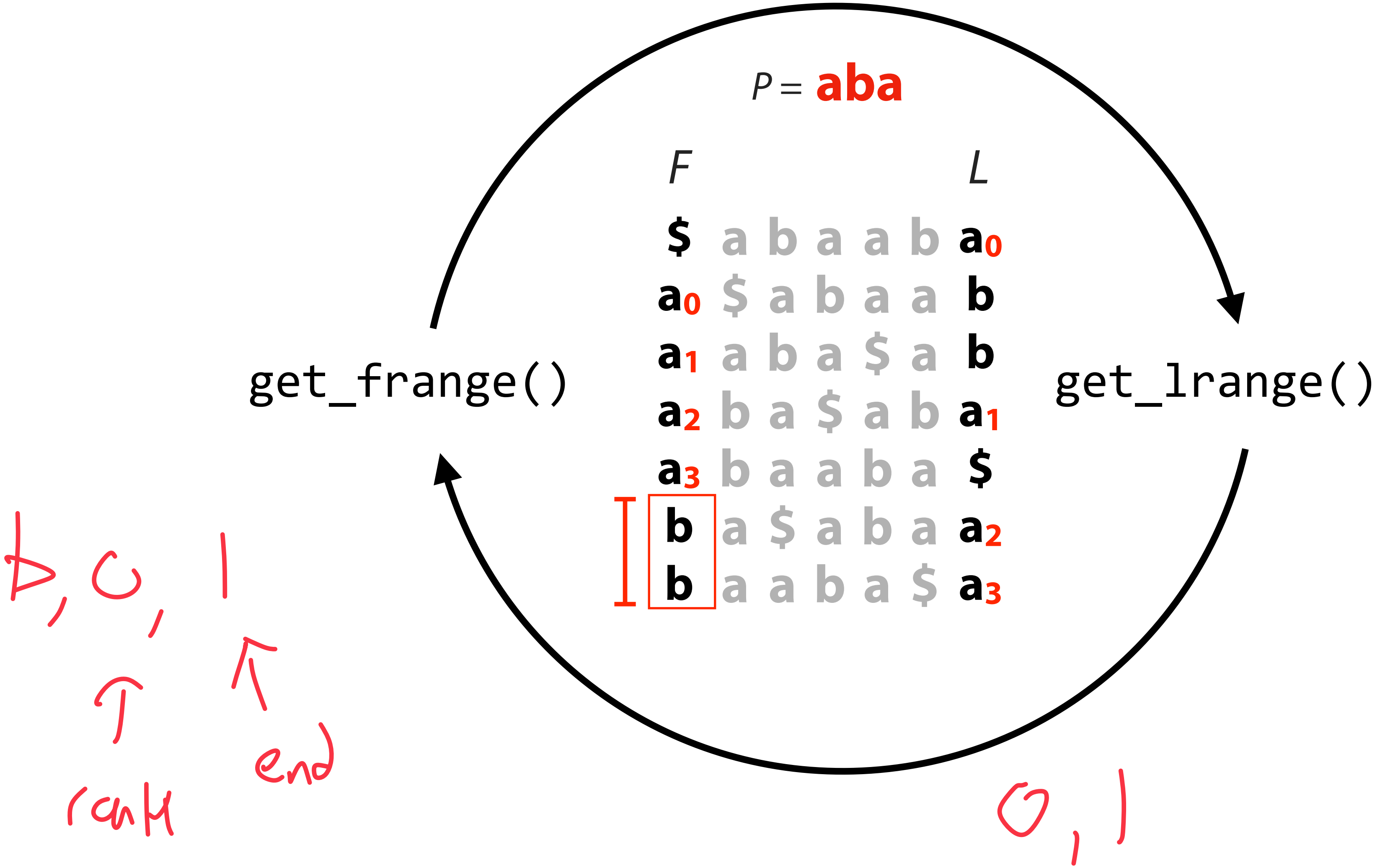
What are c, s, and e?

b, 1, 4

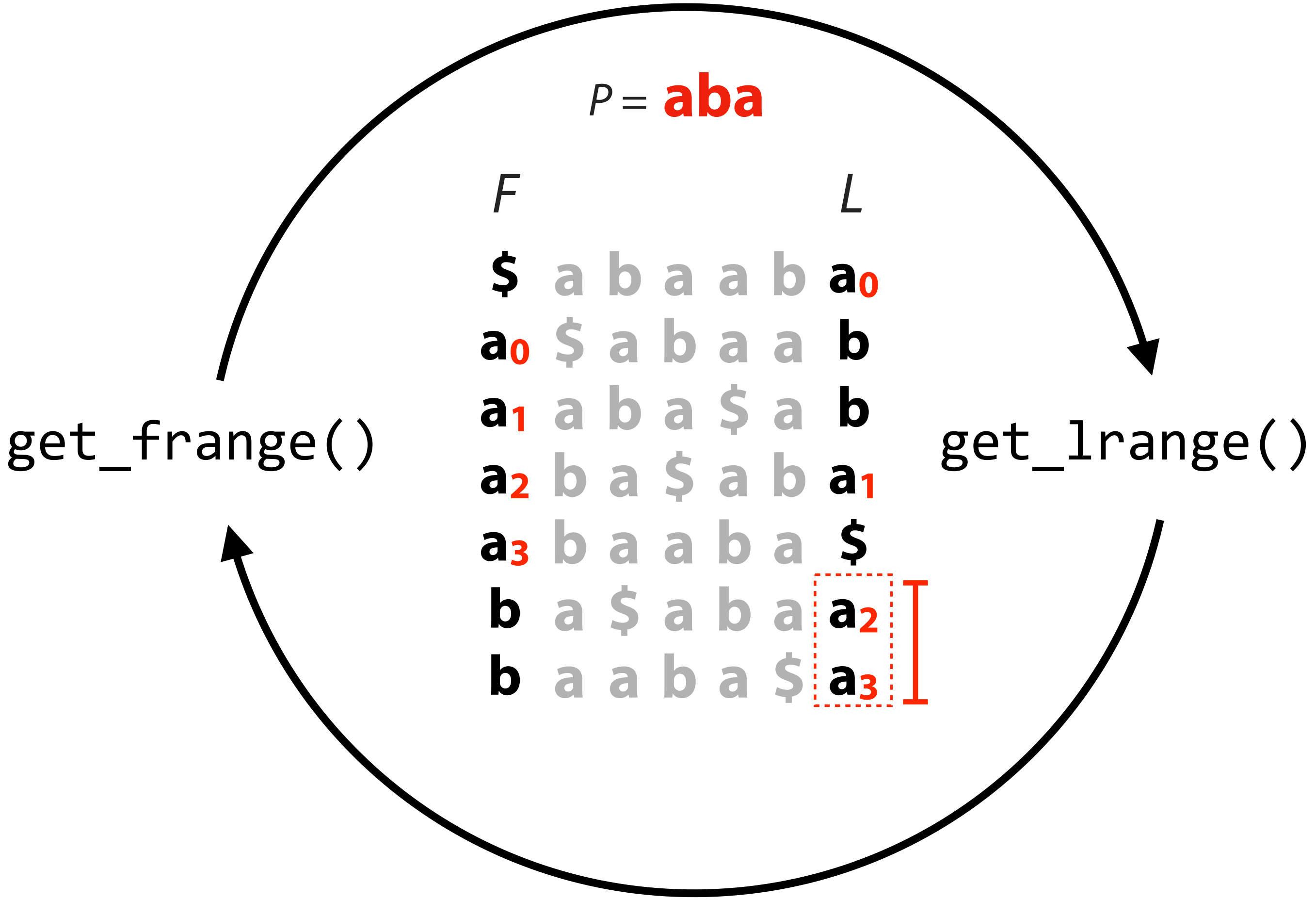
What are the output values?

Lookup occurrence table 0, 4

FM Index: Querying



FM Index: Querying



```
pair<int, int> get_frange(string c, int s, int e)
```

Input:

- **`string c`**: The char we are looking for in *F*
- **`int s`**: The starting ***rank*** value
- **`int e`**: The ending ***rank*** value

Output:

- A pair of values (index start, index end)

What are c, s, and e?

b, 0, 1

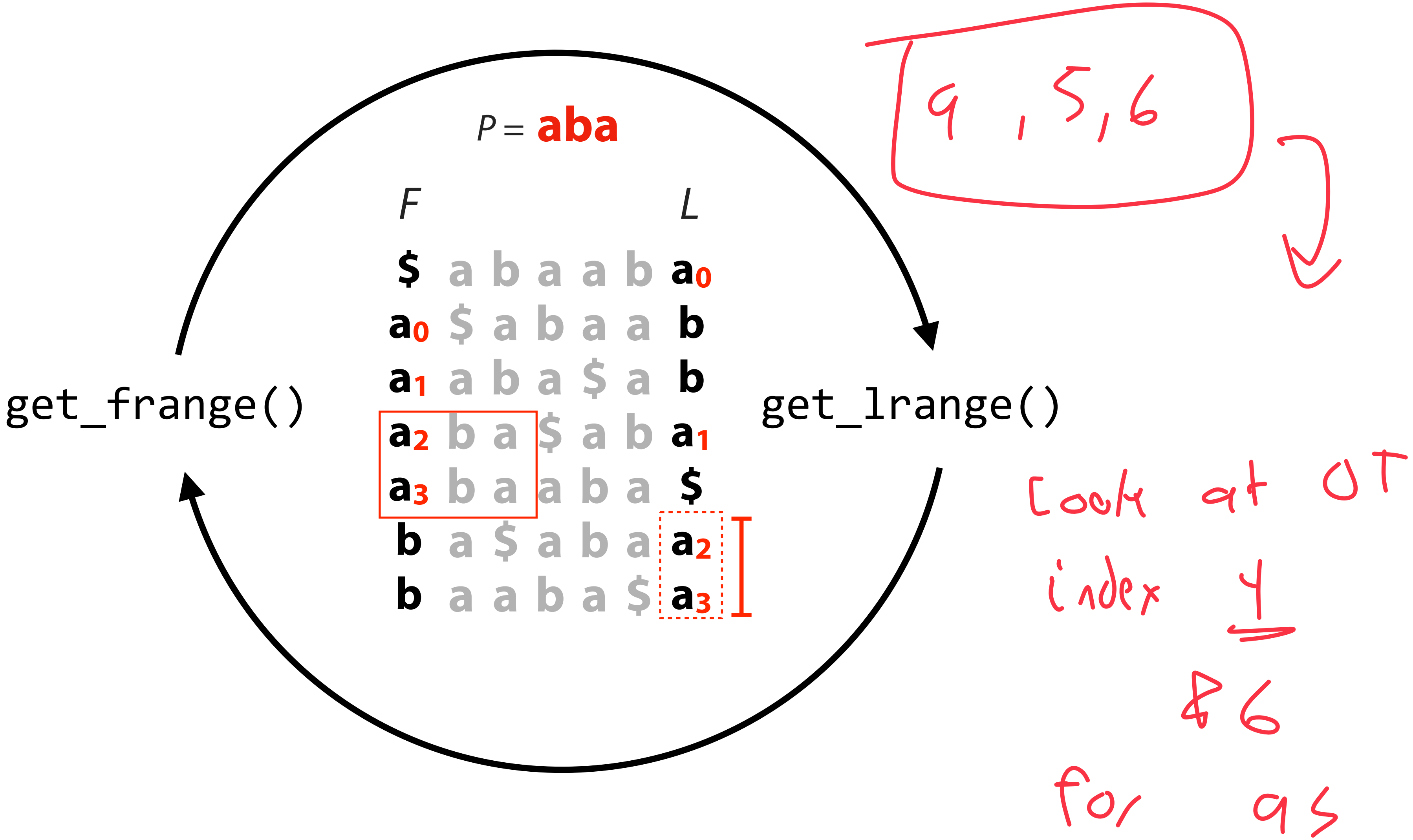
What are the output values?

[5, 6]

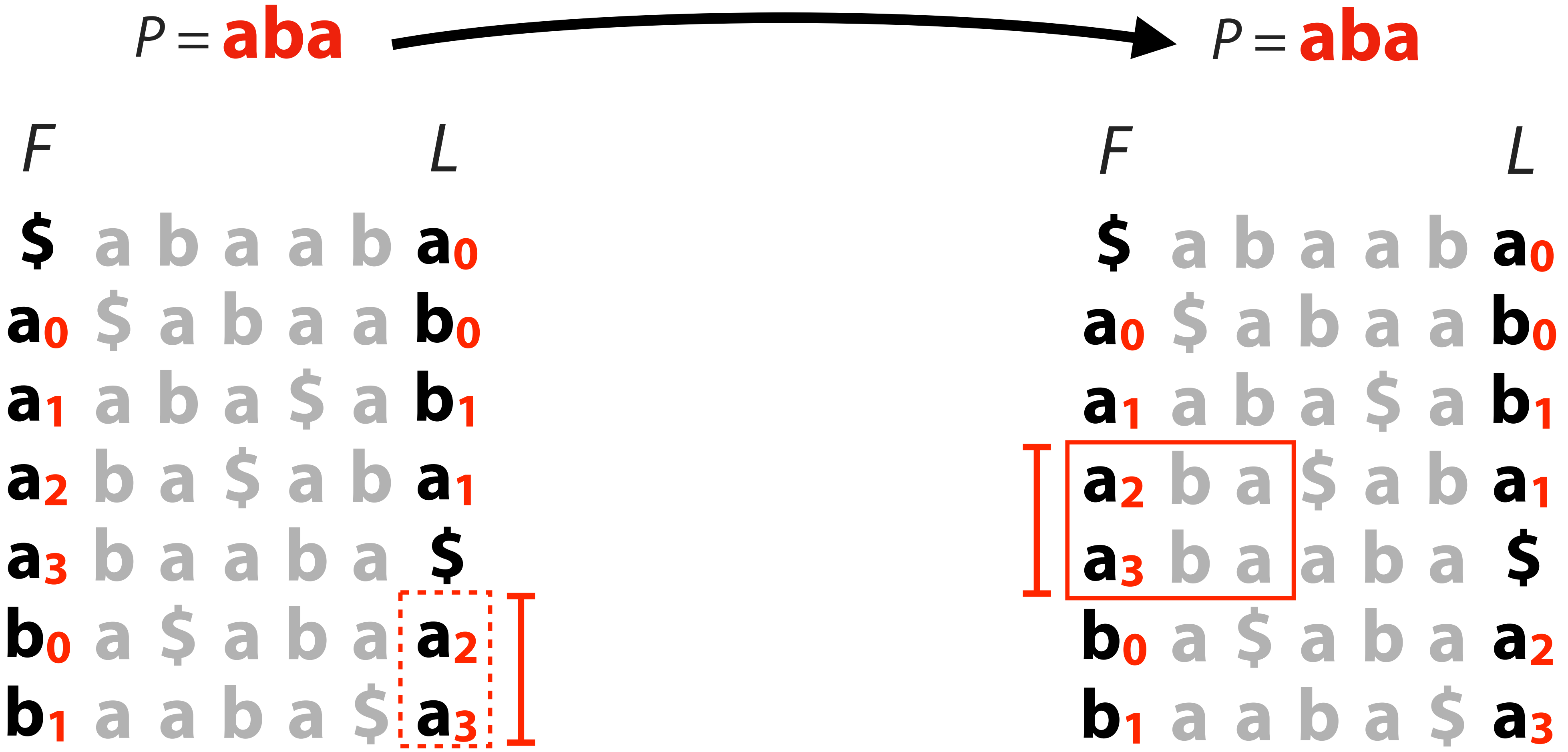
F $P =$ **aba** *L*

F						L
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

FM Index: Querying



get_lrange('a',5,6)->[2,4]



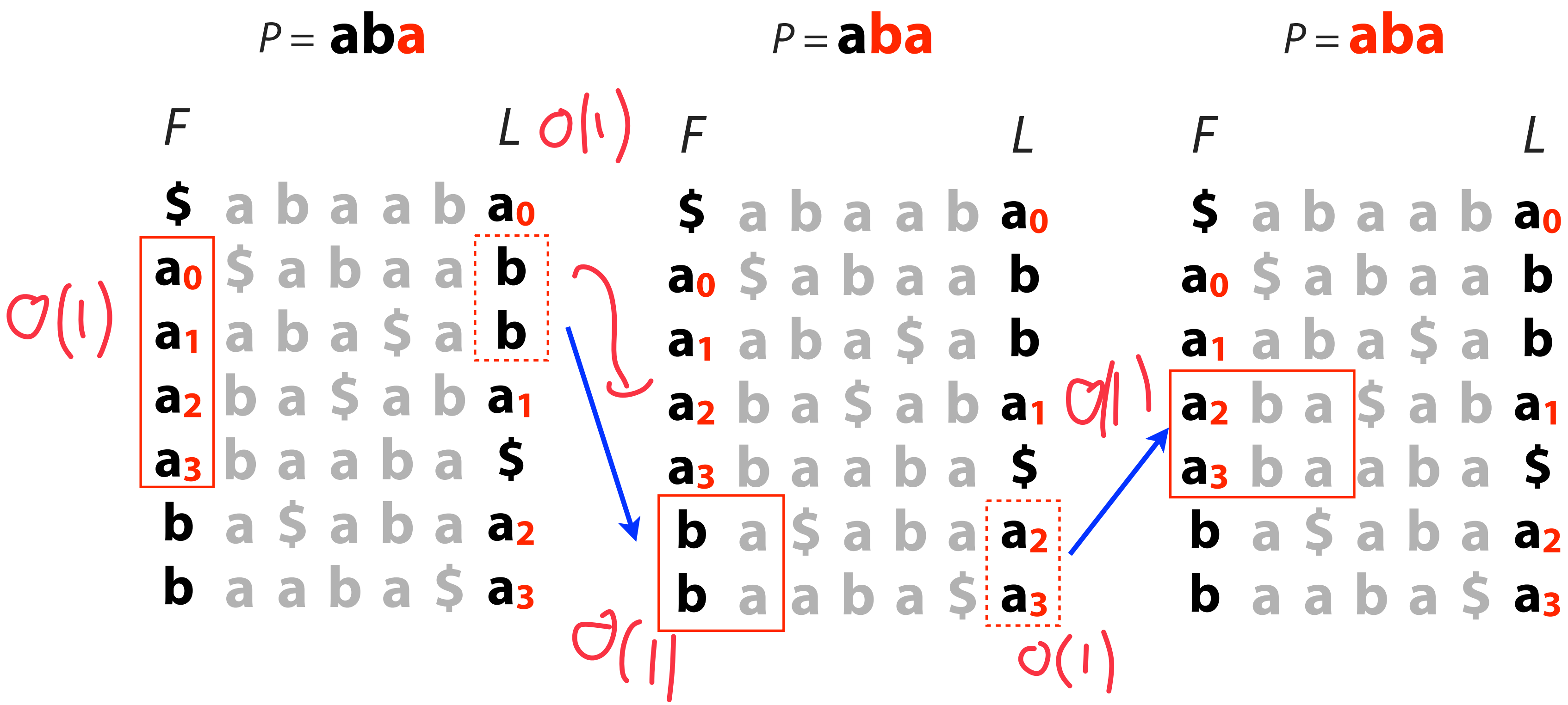
get_frange('a',2,3)->[3,4]

SA[3] = 3, SA[4] = 0 --> Return {0, 3}

FM Index

$|T| = m, |P| = n$

n times for P!



Finding all matches of *P* occurs in *T* in FM Index is O(n) time

Assignment 9: a_fmi

Learning Objective:

Construct a full FM Index

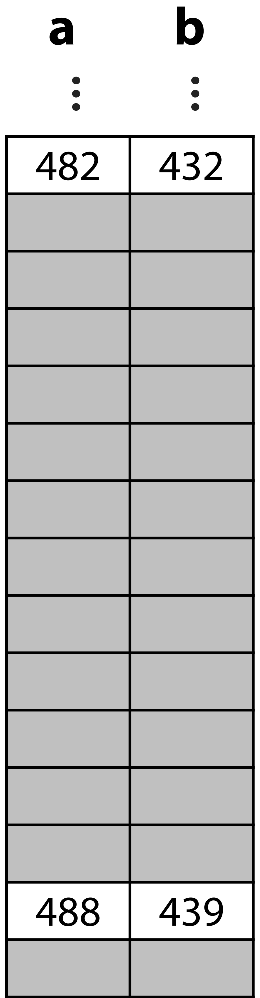
Implement exact pattern matching on a FM Index

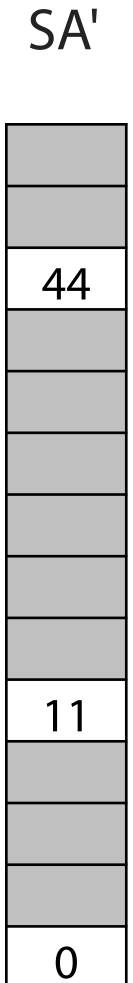
Consider: How would you modify the provided code to handle sub-sampling in the Occurrence Table (OT) or Suffix Array (SA)?

FM Index

Let a = fraction of rows we keep

Let b = fraction of SA elements we keep





FM Index consists of these, plus *L* and *F* columns

Note: suffix tree/array didn't have parameters like ***a*** and ***b***

FM Index

Components of FM Index: (blue indicates what we can adjust by changing *a* & *b*)

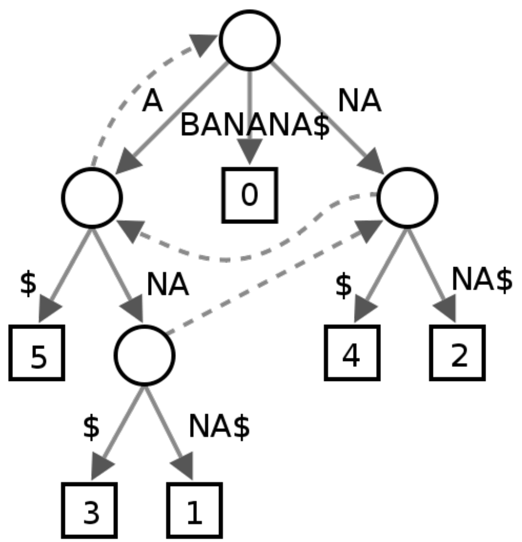
First column (*F*):	~ $\lvert\Sigma\rvert$ integers
Last column (*L*):	*m* characters
SA sample:	$m \cdot a$ integers, *a* is fraction of SA elements kept
OT Checkpoints:	$m \cdot \lvert\Sigma\rvert \cdot b$ integers, *b* is fraction of tallies kept

For DNA alphabet (2 bits / nt), *T* = human genome, ***a*** = 1/32, ***b*** = 1/128 :

First column (*F*):	16 bytes
Last column (*L*):	2 bits * 3 billion chars = 750 MB
SA sample:	3 billion chars * 4 bytes / 32 = ~ 400 MB
OT Checkpoints:	3 billion * 4 alphabet chars * 4 bytes / *128* = ~ 400 MB

Total ≈ 1.5 GB ~0.5 bytes per input char

FM Index: Small Memory Footprint



Suffix tree

≥ 45 GB

6	$
5	A$
3	ANA$
1	ANANA$
0	BANANA$
4	NA$
2	NANA$

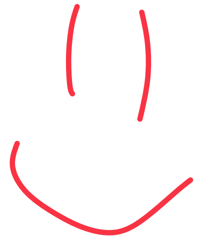
Suffix array

≥ 12 GB

$ B A N A N **A**
A $ B A N A **N**
A N A $ B A **N**
A N A N A $ **B**
B A N A N A **$**
N A $ B A N **A**
N A N A $ B **A**

FM Index

~ 1.5 GB



Suffix-Based Index Bounds

	Suffix tree	**Suffix array**	**FM Index**
Time: Does P occur?			
Time: Count k occurrences of P			
Time: Report k locations of P			
Space			
Needs T?			
Bytes per input character			

$m = |T|, n = |P|, k = \#$ occurrences of P in T

Suffix-Based Index Bounds

	Suffix tree	Suffix array	FM Index
Time: Does P occur?	$O(n)$	$O(n \log m)$	$O(n)$
Time: Count *k* occurrences of P	$O(n + k)$	$O(n \log m)$	$O(n)$
Time: Report *k* locations of P	$O(n + k)$	$O(n \log m + k)$	$O(n + k)$
Space	$O(m)$	$O(m)$	$O(m)$
Needs T?	*yes*	*yes*	*no*
Bytes per input character	>15	~4	~0.5

$m = |T|, n = |P|, k = \#$ occurrences of P in T