Data Structures

Disjoint Sets 3

CS 225
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October 20, 2023

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Learning Objectives

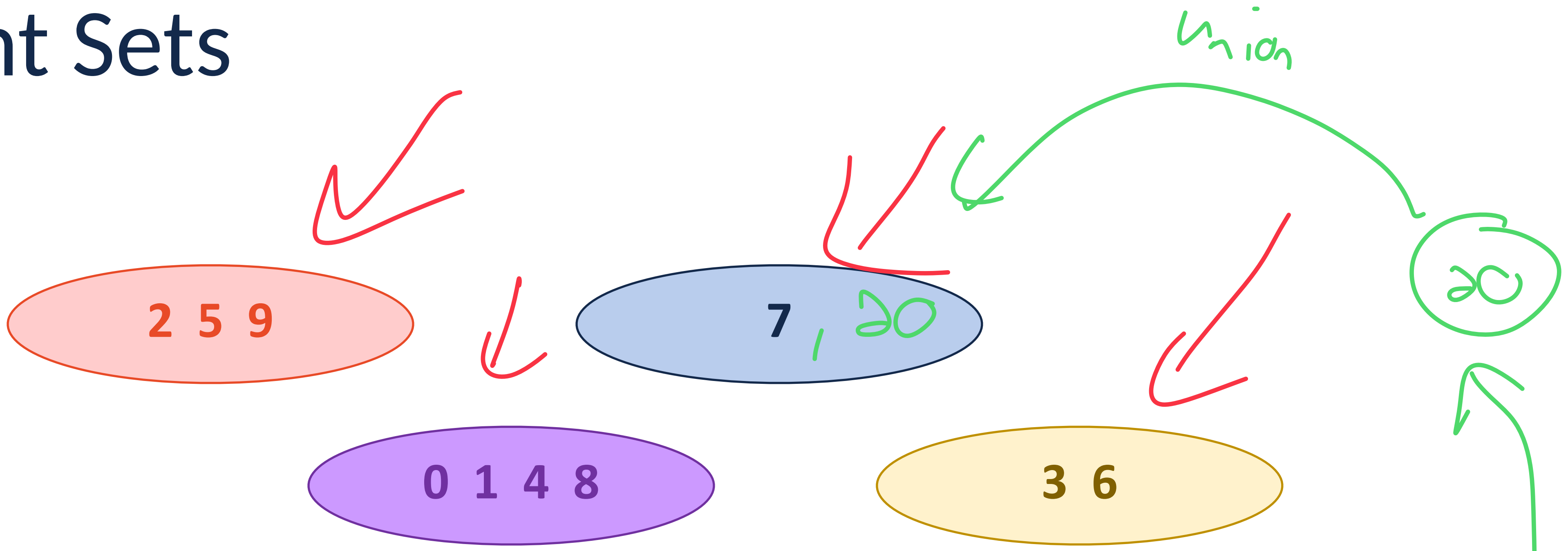
Discuss efficiency of disjoint sets

union by size
always efficient

Introduce path compression and rank

Prove efficiency of disjoint sets (again)

Disjoint Sets



Key Ideas:

- Each element exists in exactly one set.
- Every item in each set has the same representation
- Each set has a different representation

makeset(20)

Disjoint Sets Representation

We can represent a disjoint set as an array where the key is the index

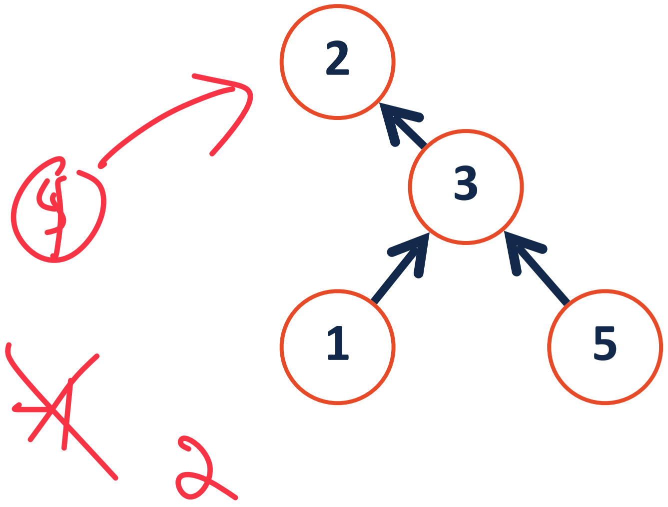
The values inside the array stores our sets as a pseudo-tree (UpTree)

-1

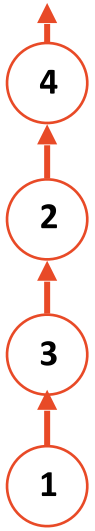
Negative values denote representative elements (the root)

All other set members store the index to a parent of the UpTree

Store height/size in the roots



Disjoint Sets – Best and Worst UpTree



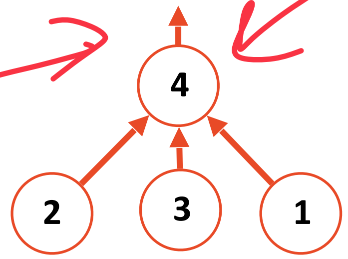
1

3

0	1	2	3	4
	3	4	2	-1

How do we design union?

Best



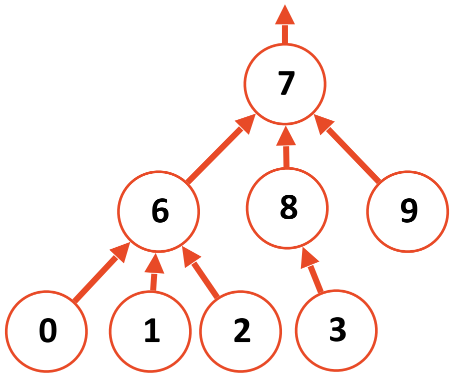
distance 1

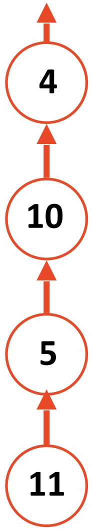
find() O(1)

0	1	2	3	4
	4	4	4	-1

Just b/c 0 is not in set

Disjoint Sets – Smart Union





- height - 1

Union by height

0	1	2	3	4	5	6	7	8	9	10	11
6	6	6	8	-4	10	7	-3	7	7	4	5

***Idea**: Keep the height of the tree as small as possible.*

Union by size

- size

0	1	2	3	4	5	6	7	8	9	10	11
6	6	6	8	-4	10	7	-8	7	7	4	5

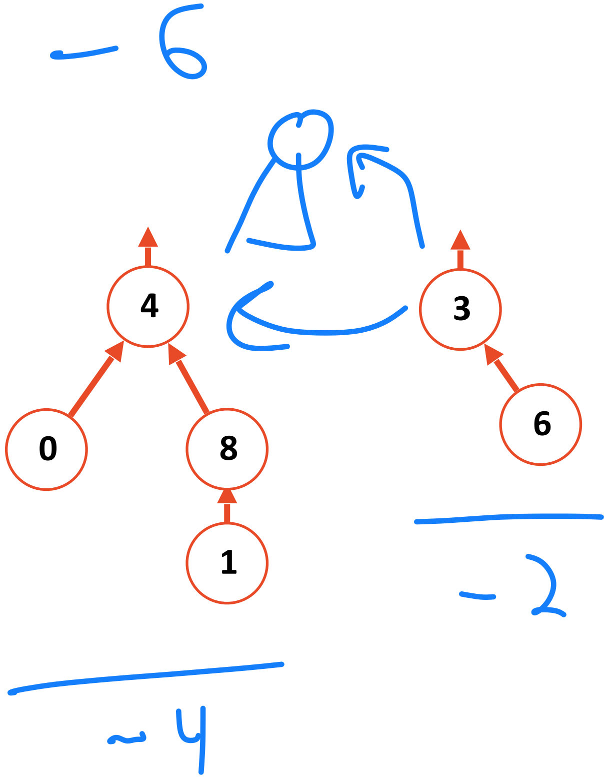
***Idea**: Minimize the number of nodes that increase in height*

Claim that both guarantee the height of the tree is: O(log n).

Disjoint Sets Union by Size

unionBySize(4, 3)

```
void DisjointSets::unionBySize(int root1, int root2) {
  int newSize = arr_[root1] + arr_[root2];

  if ( arr_[root1] < arr_[root2] ) {

    arr_[root2] = root1;

    arr_[root1] = newSize;

  } else {

    arr_[root1] = root2;

    arr_[root2] = newSize;

  }
}
```



Only ~~be~~ called on two roots

0	1	2	3	4	5	6	7	8	9
4	8		~~-2~~ 4	~~-4~~ -6		3		4	

Disjoint Sets Union by Size

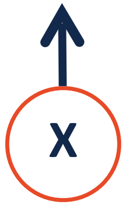
(uptree)

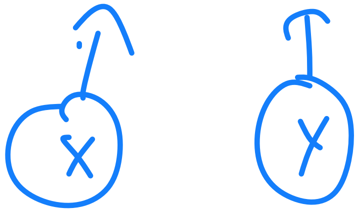
Claim: Sets unioned by size have a height of at most $O(\log_2 n)$

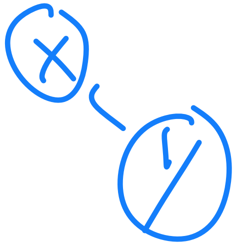
Claim: An UpTree of height **h** has nodes $\geq 2^h$

Base Case:

Base case height is 0, has one node.







$2 \geq 2^1$
nodes

vs.

$2^0 = 1$

Base case holds!

Disjoint Sets Union by Size

Let A, B be two sets being unioned

Claim: An UpTree of height **h** has nodes $\geq 2^h$

IH: Claim is true for $< i$ unions, prove for ith union.

(We have done $i - 1$ total unions and plan to do ***one*** more)

A B

A == B

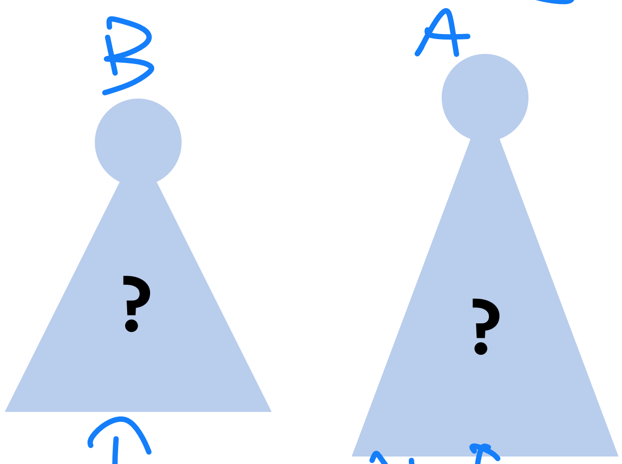
Bigger than A

n(B) $\geq$ n(A)

Case 1: h(A) < h(B)

Case 2: h(A) == h(B)

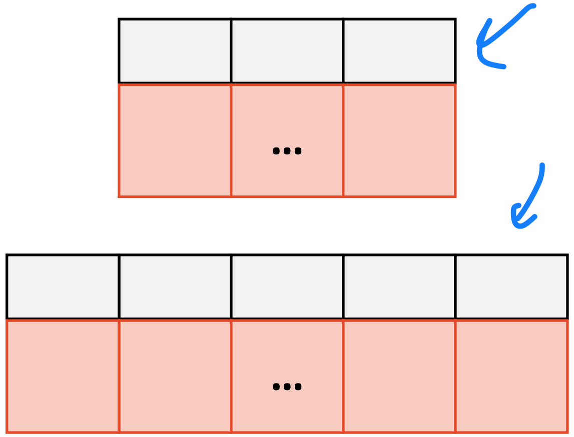
Case 3: h(A) > h(B)



T_A or B, T

A

B



Disjoint Sets Union by Size

$2^{h(B)}$

$n(B) \geq n(A)$

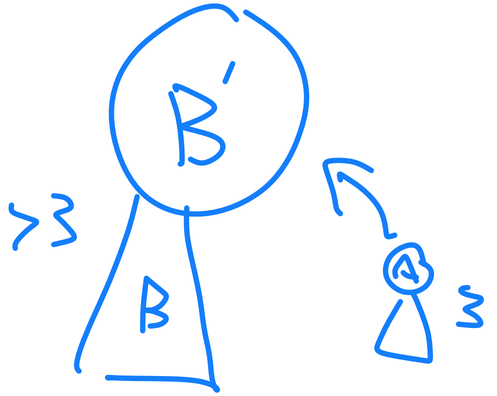
Claim: An UpTree of height **h** has nodes $\geq 2^{h}$

$h(B)$

IH: Claim is true for $< i$ unions, prove for ith union.

Case 1: height(A) < height(B)

↳ Ideal case! B/c B is our root



1) My height doesn't change!

2) By IH, both A & B are good disjoint sets

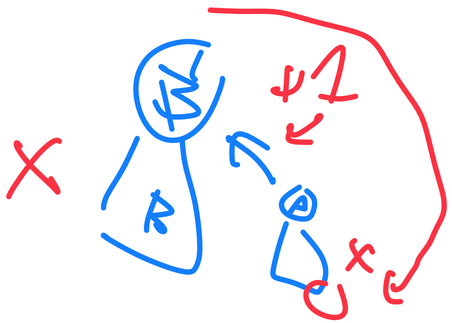
$n(B) \geq 2^{h(B)}$ and $n(A) \geq 2^{h(A)}$

$2^{h(B)} + 2^{h(A)} \geq 2^{h(B)}$

$n(B') = n(B) + n(A)$

Disjoint Sets Union by Size

$n(B) \geq n(A)$



Claim: An UpTree of height **h** has nodes $\geq 2^h$

IH: Claim is true for $< i$ unions, prove for ith union.

Case 2: height(A) == height(B)

1) My height $[h(B') = 1 + \text{height}(B)]$

By IH: $n(B') = n(B) + n(A)$

$2^{h(B)} + 2^{h(A)}$

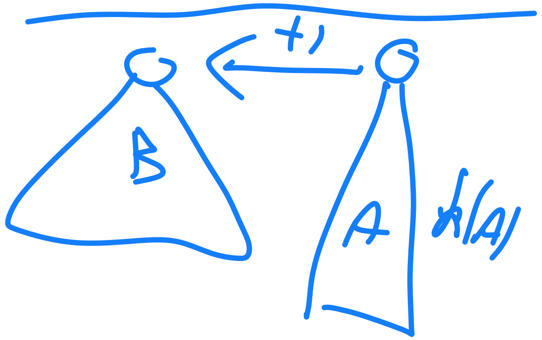
$2^{h(B)} + 2^{h(B)}$

$2 \cdot 2^{h(B)} =$

$n(B') \geq 2^{h(B)+1}$

Disjoint Sets Union by Size

$n(B) \geq n(A)$



Claim: An UpTree of height **h** has nodes $\geq 2^h$

IH: Claim is true for $< i$ unions, prove for ith union.

Case 3: height(A) > height(B)

know $n(A) \geq 2^{h(A)}$

$n(B) \geq$

$n(B) = n(A) + n(B) \geq 2n(A)$

$2 \cdot 2^{h(A)}$

$2^{h(A)+1}$

$n(B') \geq 2^{h(A)+1}$

Disjoint Sets Union by Size

$n(B) \geq n(A)$

Proven: An UpTree of height **h** has nodes $\geq 2^h$

IH: Claim is true for $< i$ unions, prove for ith union.

Each case we saw we have $n \geq 2^h$.

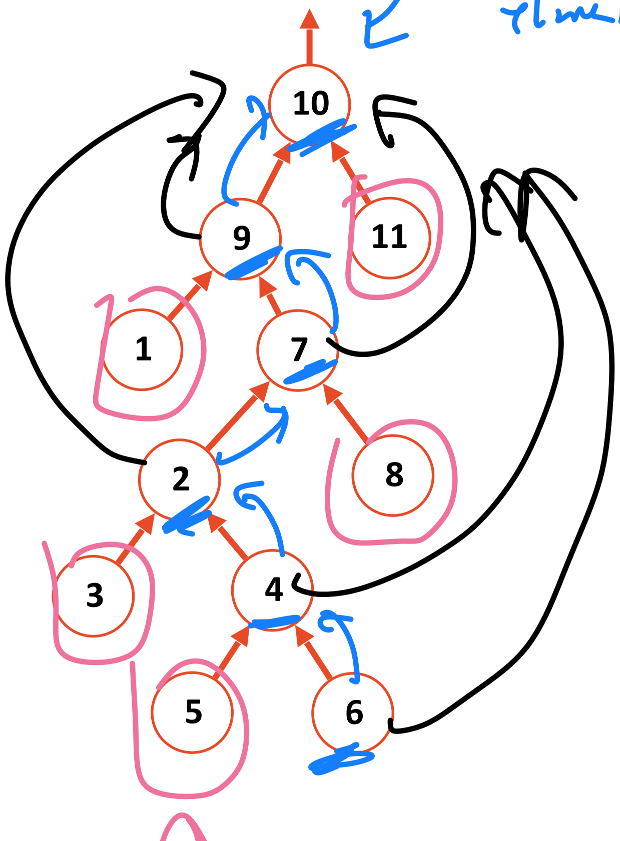
$h \approx O(\log n)$

An uptree unioned by size still has good height

Path Compression

Find(6)
↳ 10 = 10

canonical element



Not touched!

find (6) // 10
↳ find(4) // 10
↳ .... find(10)

find(2) // 10

return 10
return 10 . . .
returns 10

when we recurse up, on the way back set all nodes to point to root directly

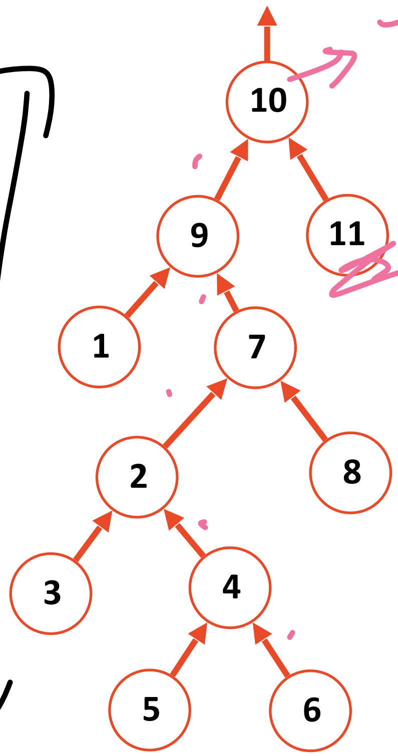
Path Compression

Find(6)

Assume log(n)

The more find() the closer we get

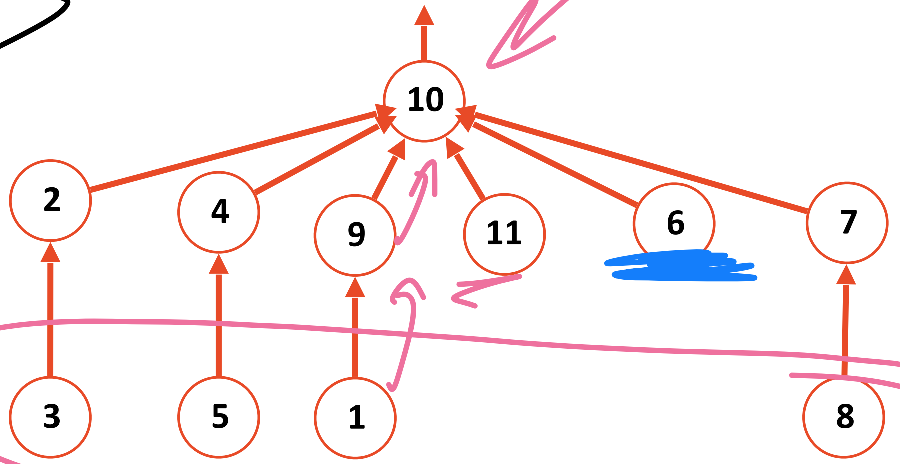
~O(1)



O(h)

-6

O(1) · h times



-6

No more work than find() was done!

find(1)

Intuitively: I only do work once

Nothing else changes

O(h)

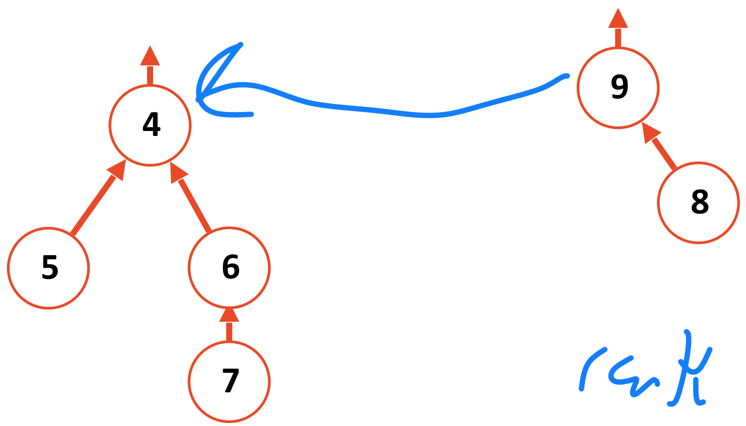
Disjoint Sets – Union by Rank (not height!)

union is max(A, B)+1

rank = 1

rank : 0 0 0 0

[zero]



rank 2

rank 3

Height + Path compression == lies!

-3

4

X

We will also store as - rank

0	1	2	3	4	5	6	7	8	9
				-3					-2

Union by Rank (Not Height)

motivated by Path compression

The change: New UpTrees have rank = 0

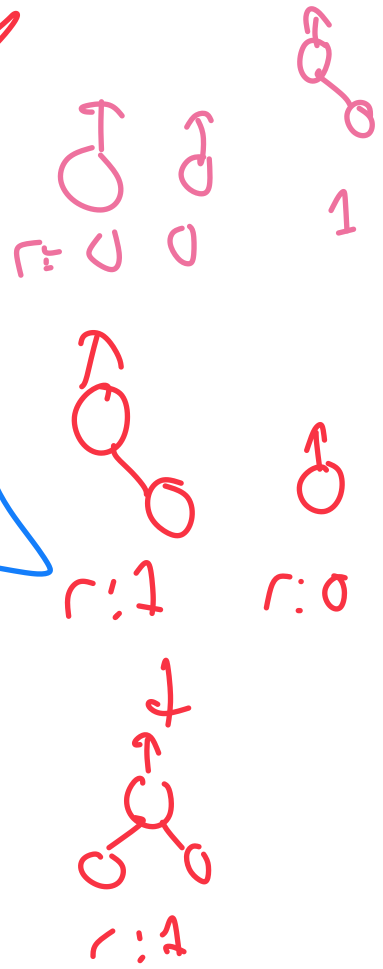
Let A, B be two sets being unioned. If:

rank(A) == rank(B): The merged UpTree has rank + 1

rank(A) > rank(B): The merged UpTree has rank(A)

rank(B) > rank(A): The merged UpTree has rank(B)

This is identical to height (with a different starting base)!



Union by Rank

Claim: An UpTree of rank **r** has nodes $\geq 2^r$.

Base Case:

Inductive Step: IH holds for all UpTrees up to $k < r$

Exercise for viewers

Try solving yourself before seeing answer (next slide)!

Union by Rank - Proof

Much like before we will show that in a tree with a root of rank r there are $nodes(r) \geq\ 2^r$

Base Case: UpTree of rank = 0 has 1 node $2^0 = 1$

Inductive Hypothesis: for all trees of ranks $k,\ k < r,\ \ nodes(k) \geq 2^k$

A root of rank r is created by merging two trees of rank $r - 1$

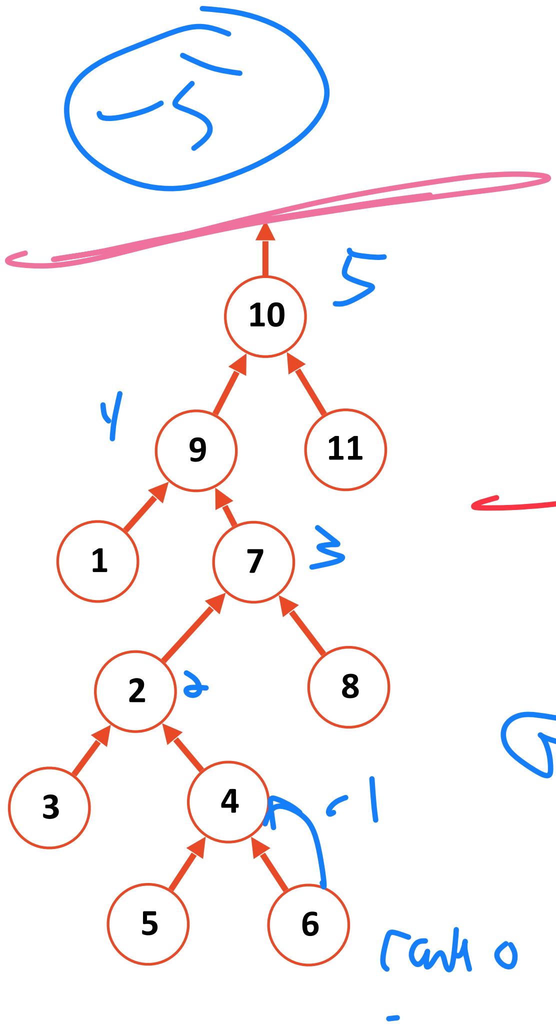
by IH each of those trees have $nodes(r-1) \geq 2^{r-1}$

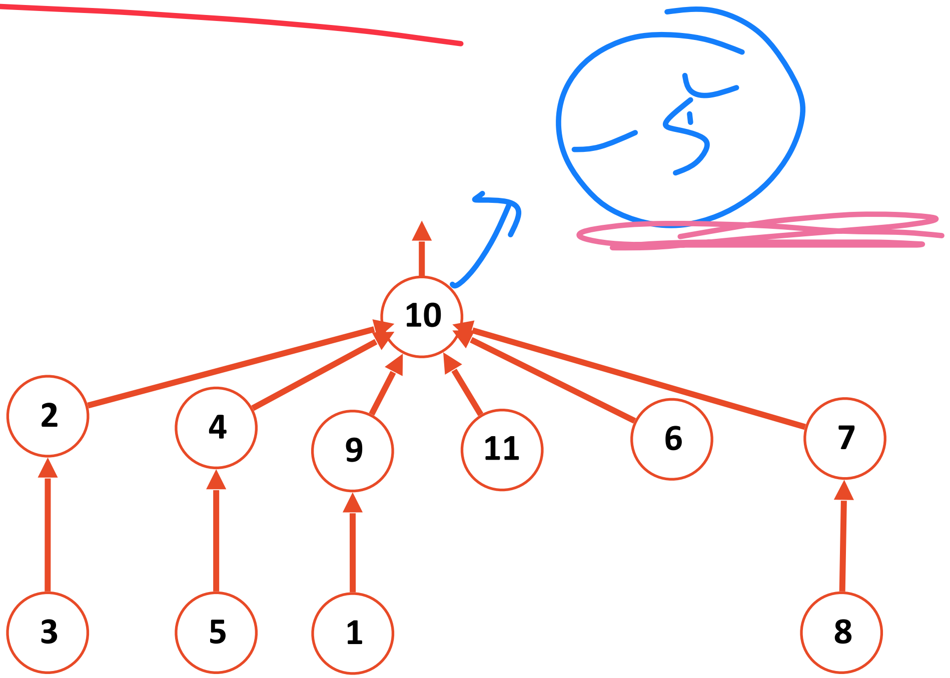
so, tree a of rank r has $nodes(r) \geq 2 \times 2^{r-1} \geq 2^r$

Taking the inverse, we get a height of $O(\log(n))$

Union by Rank w/ Path Compression

How does rank w/ path compression affect our runtime?





Find(i) → O(log n) worst

Union by Rank w/ Path Compression

1. Rank only changes for roots and can only increase (unlike height!)

↳ canonical elements

$r == r \Rightarrow r+1$

This is why union is O(1)

2. For all non-root nodes x, **rank(x) < rank(parent(x))**

3. If parent(x) changes, then our new parent has larger rank.

$r-1, r-1 \Rightarrow r$ $\quad r_1 < r \Rightarrow r$ $\quad r, r \Rightarrow r+1$

Union by Rank w/ Path Compression

4. min(nodes) in a set with a root of rank r has $\geq 2^r$ nodes.

↳ proof I skipped

$n \geq 2^r$

restating 4

5. Since there are only n nodes the highest possible rank is $\lfloor log\ n \rfloor$.

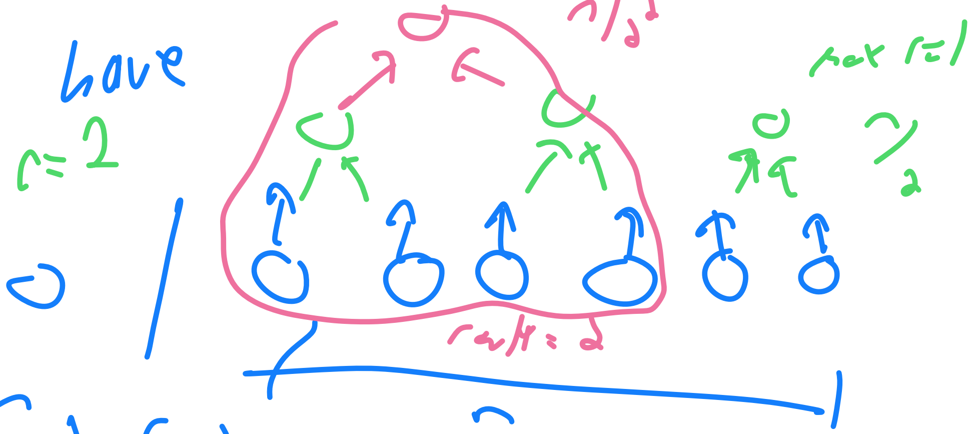
$h \sim O(\log n)$

Union by Rank w/ Path Compression

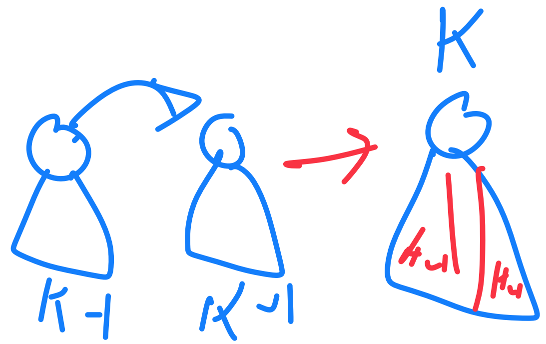
6. For any integer r, there are at most $\dfrac{n}{2^r}$ nodes of rank r.

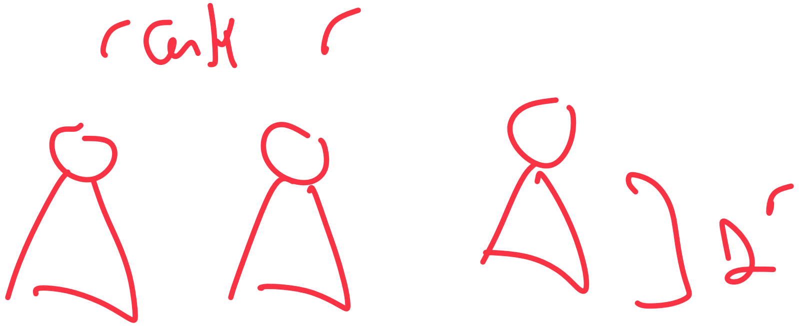
Intuitively: The larger my rank, the more nodes stored below me, the fewer total nodes I can have

Base case: $r = 0 \rightarrow n$ nodes of rank 0



For any k, $r = k$ only when we union $(k-1) \times (k-1)$





$n/2^r$ is the max # I can have of rank r

Amortized Time (Rank w/ Path Compression)

will be different

For **n** calls to makeSets() [**n items**] and **m** find() calls the max work is…

?? amount of union

Built best uptree

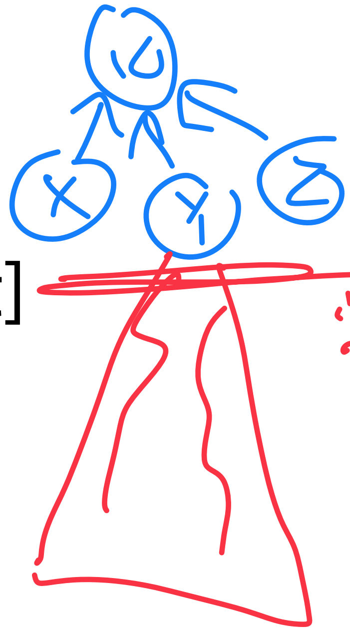
This gives us a more accurate picture since each find can make our search a faster!

Two cases of find():

canonical element

1. We search for root [or a node whose parent is root]

$O(1)$



Path compression

2. We search for a node where neither above apply.

Proof comes in

Do this Monday

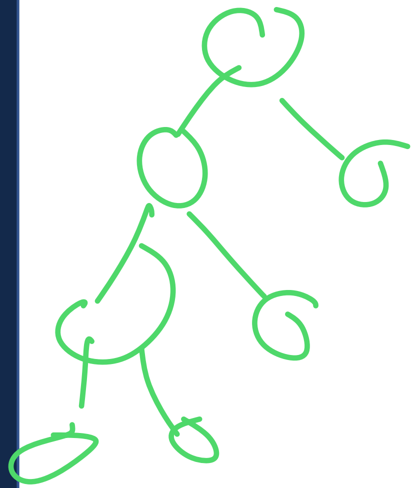
Amortized Time (Rank w/ Path Compression)

Put every non-root node in a bucket by rank!

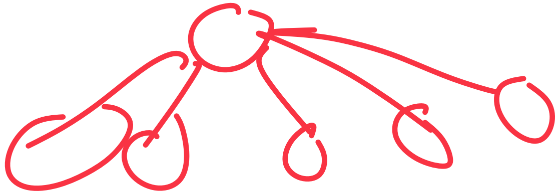
Ranks	Bucket
0	0
1	1
2 - 3	2
4 - 15	3
16 – 65535	4
65536 – 2^{65536}-1	5

Structure buckets to store ranks $[r,\ 2^r - 1]$

Max work:

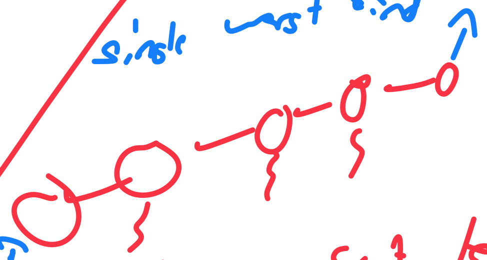


After n steps have explored every path



I'm missing, I have ~~bad~~

single worst find



find()

b/c we can't visit have direct LL

b/c $n \geq 2^h$

Iterated Logarithm Function (log^*n)

log^*n is piecewise defined as

0 if $n \leq 1$

otherwise

$1 + log^*(\log n)$

Amortized Time (Rank w/ Path Compression)

Let $|B_r|$ be the size of the bucket with min rank r.

What is $max(|B_r|)$?

Ranks	Bucket
0	0
1	1
2 - 3	2
4 - 15	3
16 – 65535	4
65536 – 2^{65536}-1	5

Amortized Time (Rank w/ Path Compression)

The work of **find(x)** is the steps taken on the path from a node x to the root (or immediate child of the root) of the UpTree containing x

We can split this into two cases:

Case 1: We take a step from one bucket to another bucket.

Case 2: We take a step from one item to another inside the same bucket.

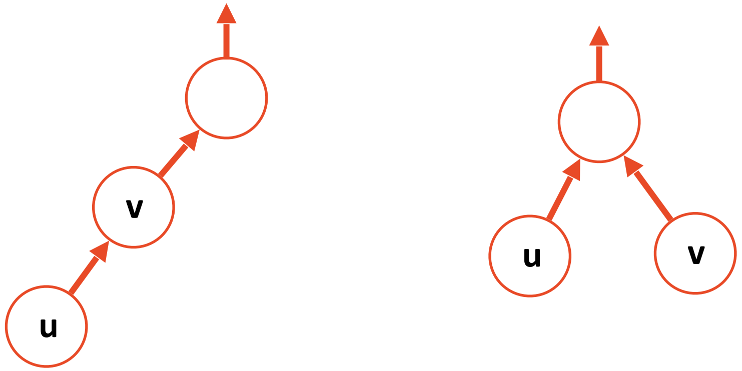
Amortized Time (Rank w/ Path Compression)

Case 2: We take a step from one item to another *inside* the same bucket.

Let's call this the step from **u** to **v**.

Every time we do this, we do path compression:

We set parent(u) a little closer to root



How many total times can I do this for each **u** in $|B_r|$?

How many nodes are in $|B_r|$?

Final Result

For **n** calls to makeSets() [**n items**] and **m** find() calls the max work is:

Even Better

In case that still seems too slow tightest bound is actually

$$\Theta(m\ \alpha(m, n))$$

Where $\alpha(m, n)$ is the inverse Ackermann function which grows much slower than log*n.

Proof well outside this class.

Randomized Algorithms

A **randomized algorithm** is one which uses a source of randomness somewhere in its implementation.

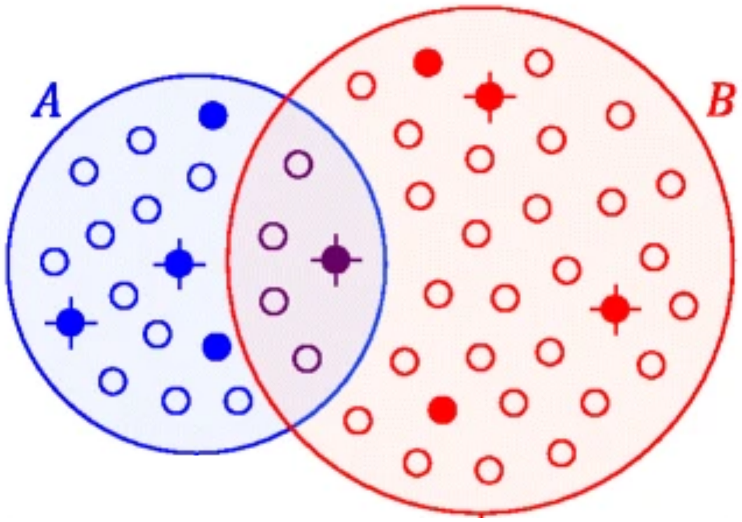
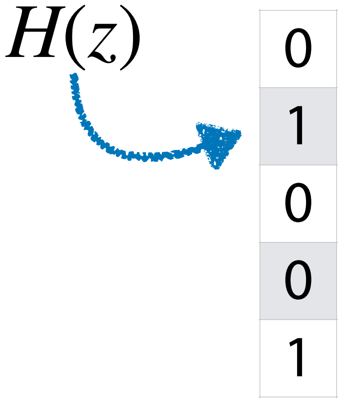
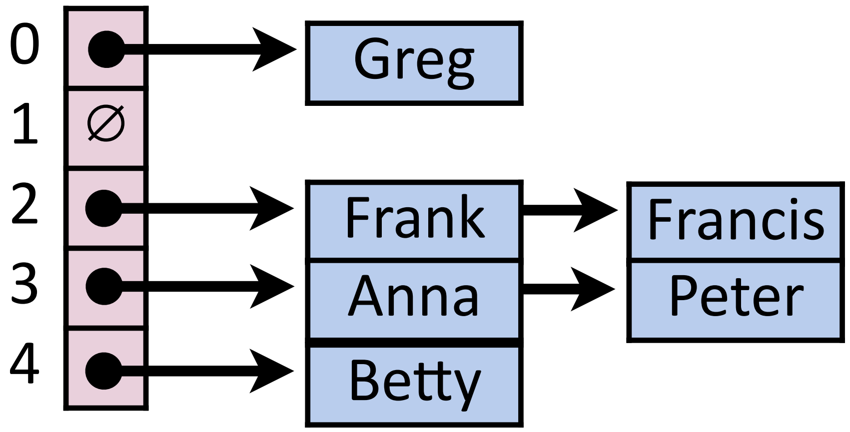


Figure from Ondov et al 2016





$H(x)$	0	2	1	0	0	4	0	2	0	6
$H(y)$	1	0	2	3	1	0	3	4	0	1
$H(z)$	2	1	0	2	0	1	0	0	7	2