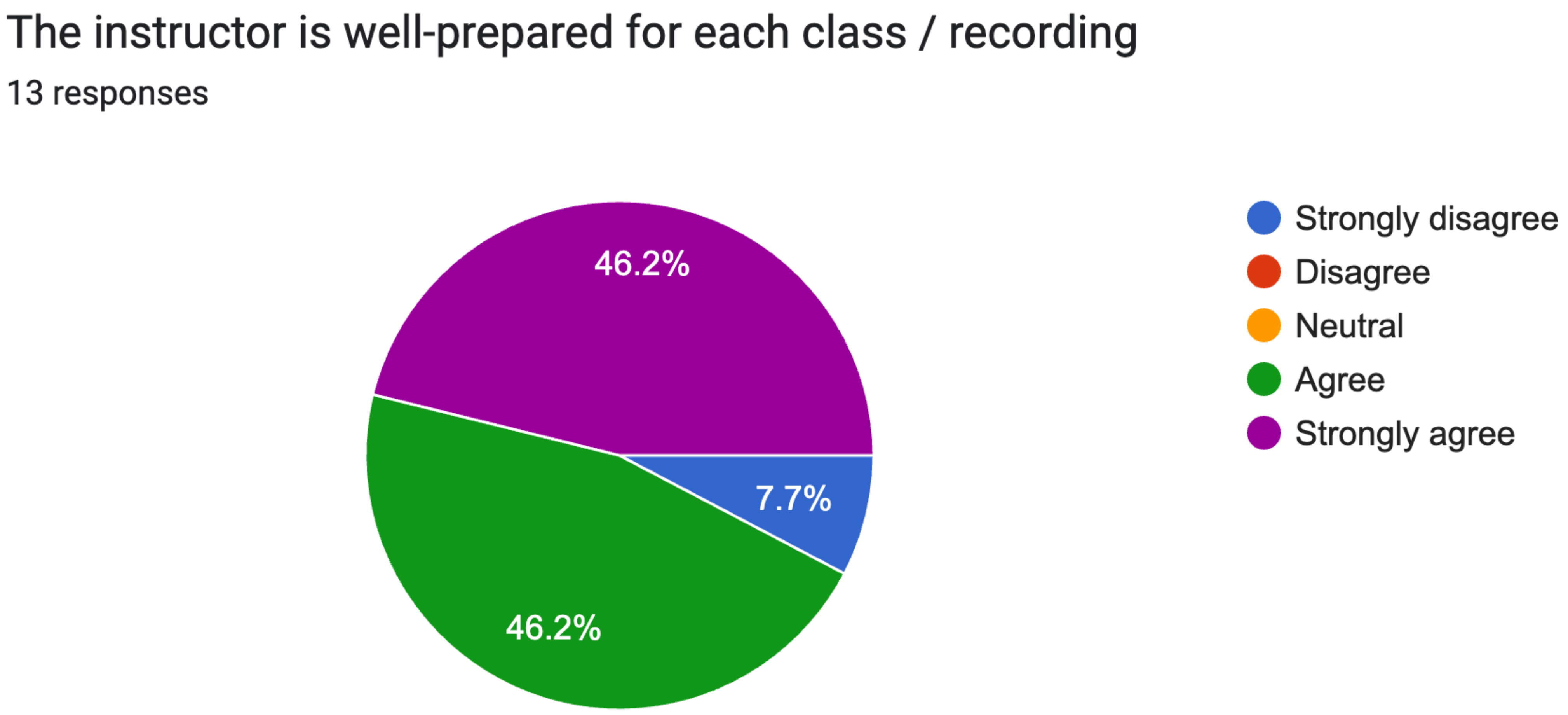
String Algorithms and Data Structures
Burrows-Wheeler Transform

CS 199-225
Brad Solomon

October 24, 2022

UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN
Department of Computer Science

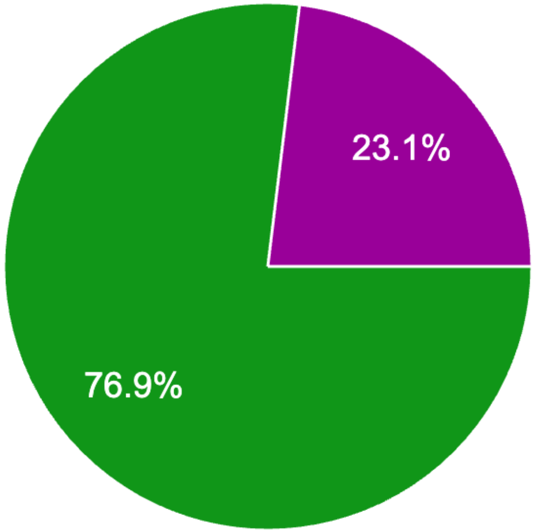
Informal Early Feedback



Informal Early Feedback

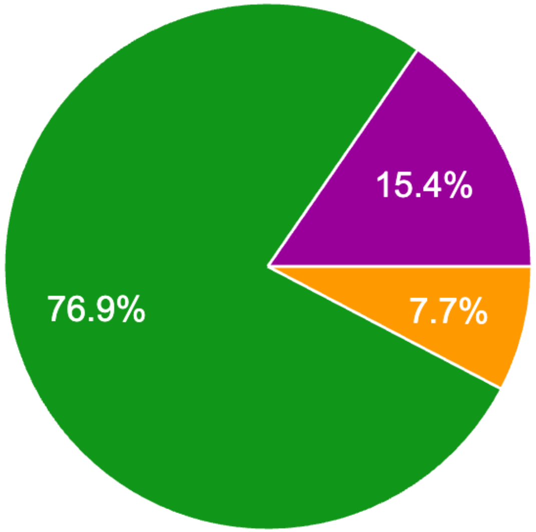
I feel that I can actively participate in lecture

13 responses



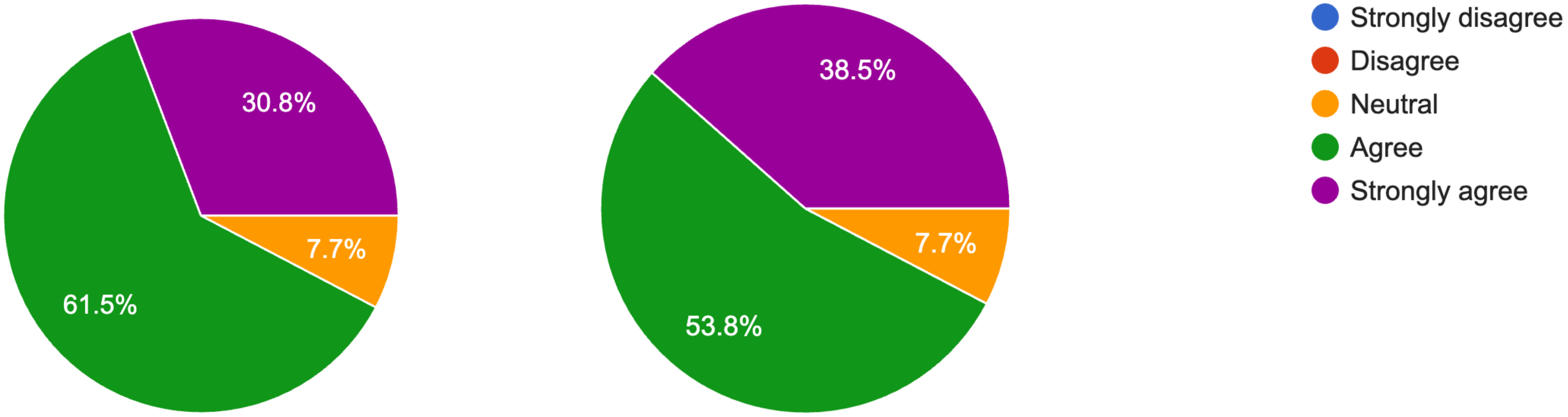
I feel that I can actively participate in class in general

13 responses



Informal Early Feedback

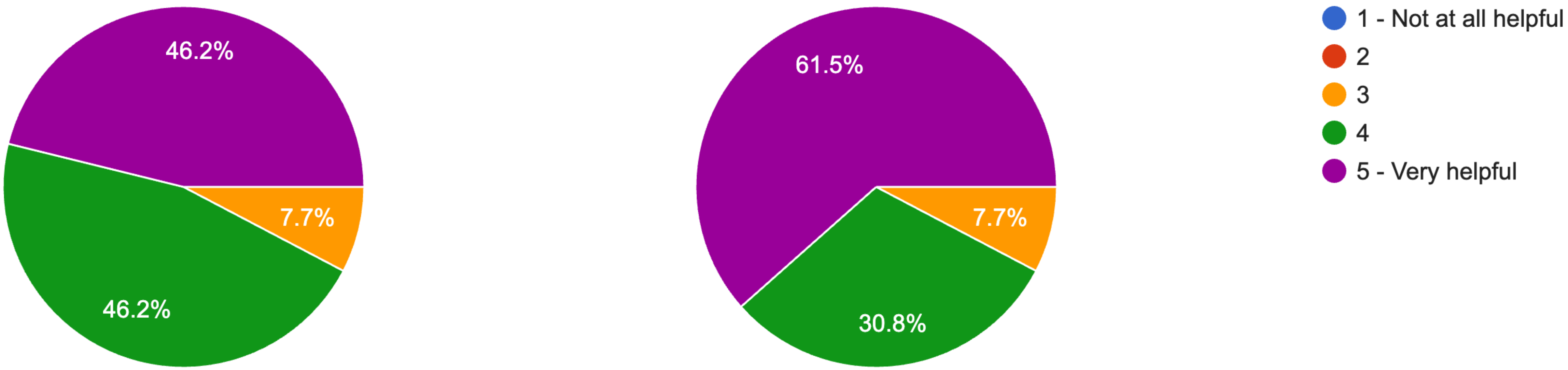
I receive helpful and complete answers to my questions



During lecture

Outside lecture

Informal Early Feedback



Lecture helpfulness

Assignment helpfulness

Informal Early Feedback

The discord is pretty useful, as the instructor often responds to answer questions.

It's hard to decide between the lectures and assignments. Both have been instrumental.

recorded lectures / slides [are the most helpful]

Getting some of the hidden test cases or charComps test cases is very difficult. I would like it more if these test cases were given or revealed, though I understand if this isn't possible.

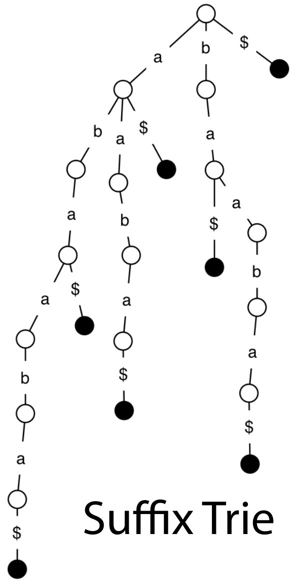
Maybe spend more time on big O analysis. It's really confusing sometimes.

I wish we could move back to in-person lectures.

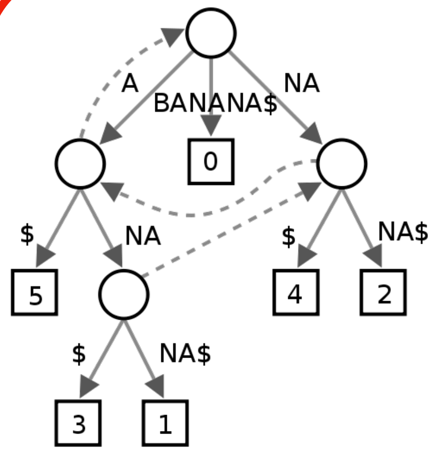
Exact pattern matching *w/ indexing*

There are many data structures built on ***suffixes***

We have now seen both of these data structures



Suffix Trie



Suffix Tree

6	$
5	A$
3	ANA$
1	ANANA$
0	BANANA$
4	NA$
2	NANA$

Suffix Array

$ B A N A N A
A $ B A N A N
A N A $ B A N
A N A N A $ B
B A N A N A $
N A $ B A N A
N A N A $ B A

FM Index

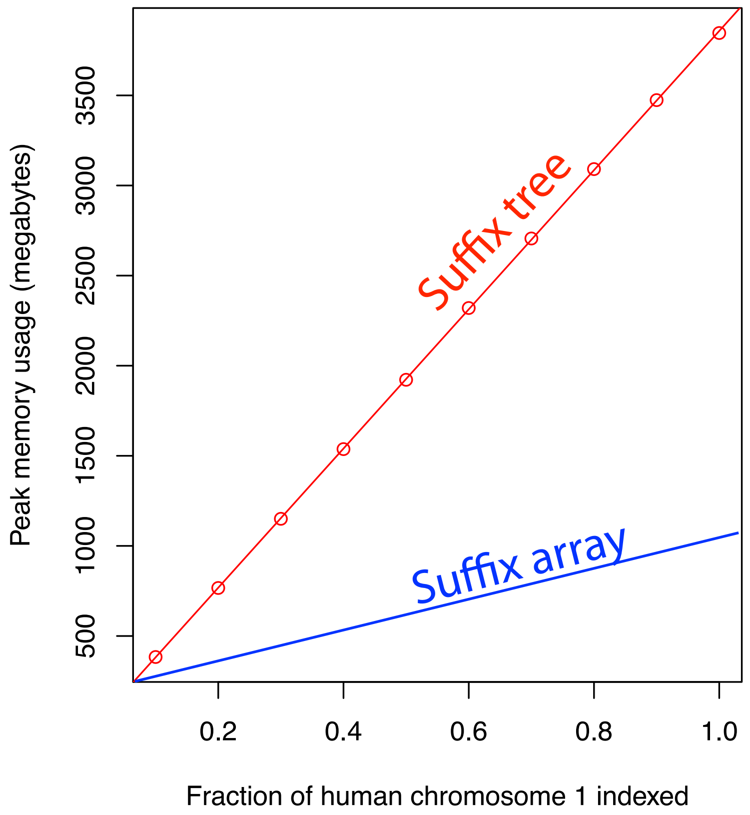
Exact pattern matching *w/ indexing*

	Suffix tree	**Suffix array**
Time: Does P occur?		
Time: Report *k* locations of P		
Space		

$m = |T|, \; n = |P|, \; k = \#$ occurrences of P in T

Suffix tree vs suffix array: size

The suffix array has a smaller constant factor than the tree



Suffix tree: ~16 bytes per character

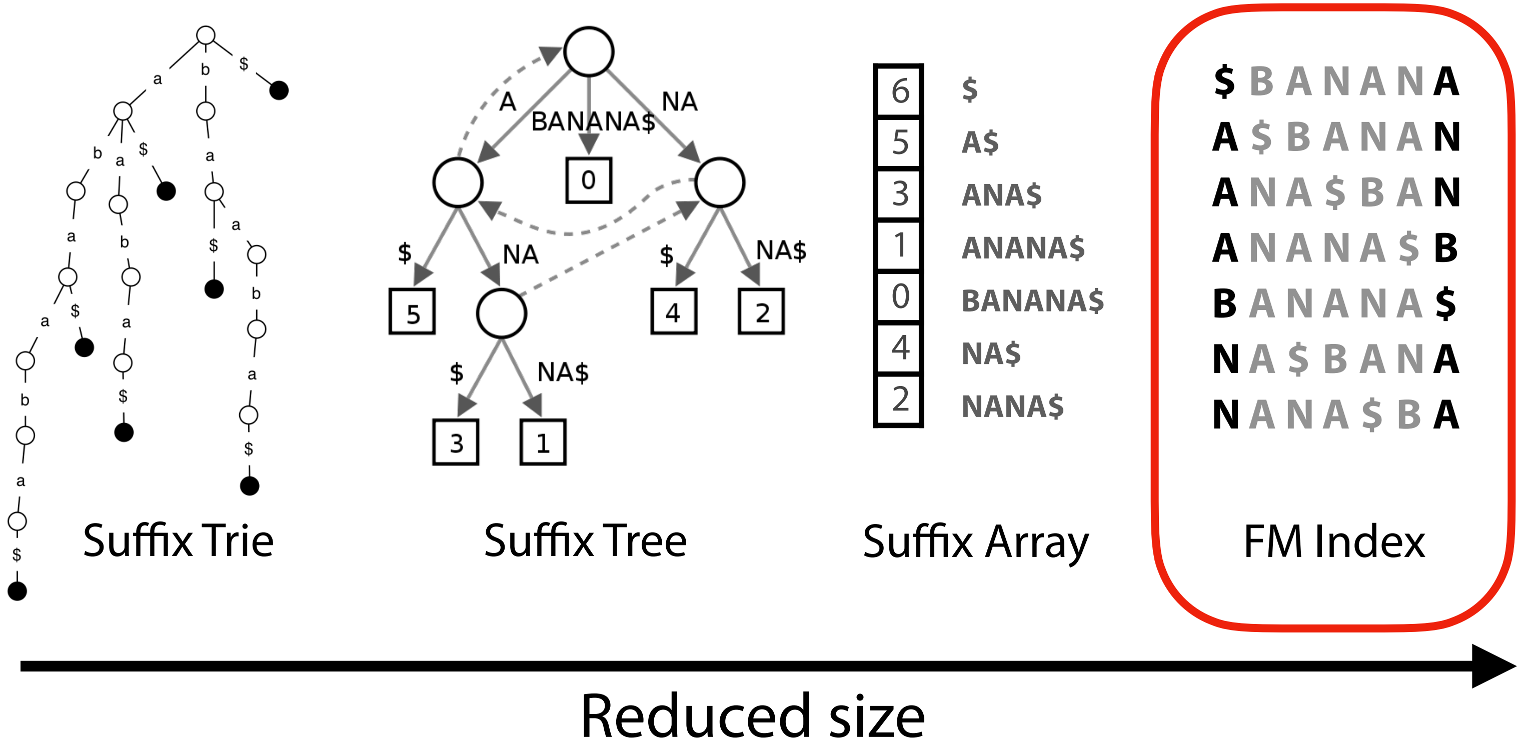
Suffix array: ~4 bytes per character

Raw text: 2 bits per character

Exact pattern matching *w/ indexing*

There are many data structures built on ***suffixes***

The FM index is a compressed self-index (smaller* than original text)!



Exact pattern matching *w/ indexing*

The basis of the FM index is a *transformation*

B A N A N A $

A N N B $ A A

Burrows-Wheeler Transform

Reversible permutation of the characters of a string

T		BWT(T)
B A N A N A $	⟷	A N N B $ A A

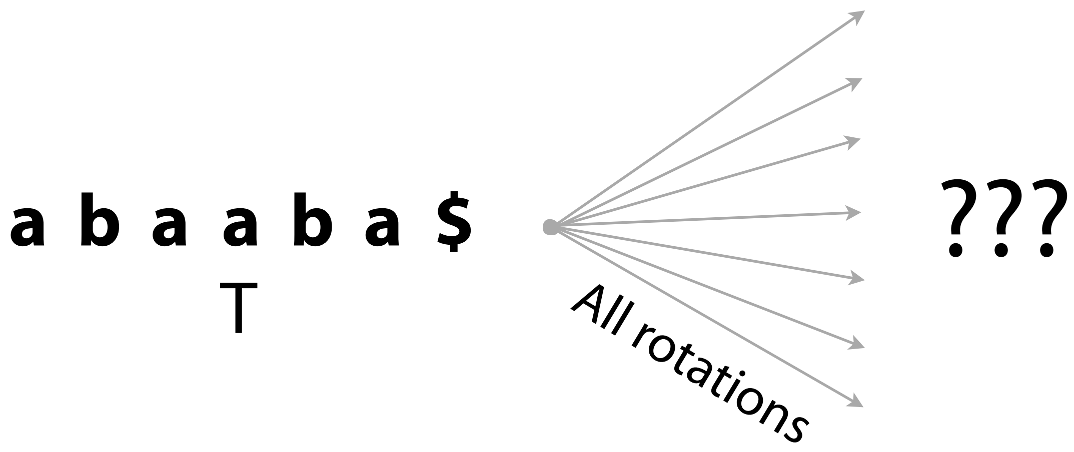
1) How to encode?

2) How to decode?

3) How is it useful for search?

Burrows-Wheeler Transform

Reversible permutation of the characters of a string



Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Text rotations

A string is a 'rotation' of another string if it can be reached by wrap-around shifting the characters

a b c d e f $

b c d e f $ a

c d e f $ a b

d e f $ a b c

e f $ a b c d

f $ a b c d e

$ a b c d e f

(after this they repeat)

Text Rotations

A string is a ‘rotation’ of another string if it can be reached by wrap-around shifting the characters

Which of these are rotations of ‘ABCD’?

A) BCDA

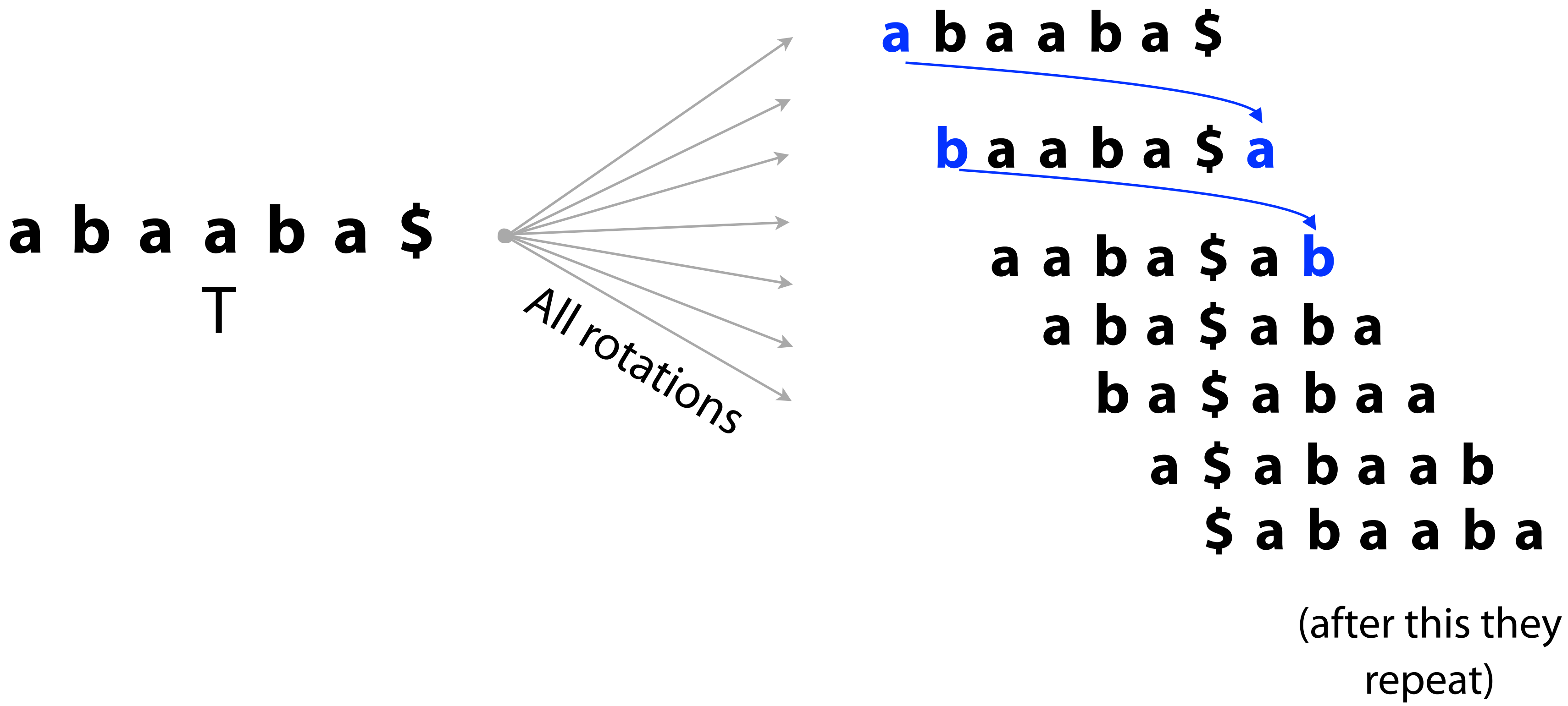
B) BACD

C) DCAB

D) CDAB

Burrows-Wheeler Transform

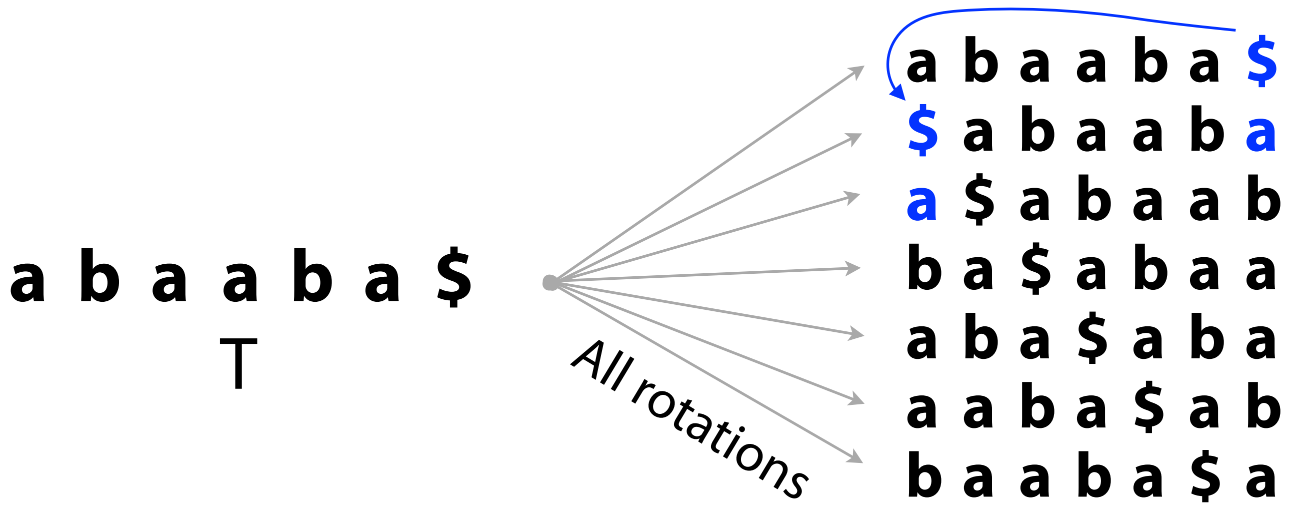
Reversible permutation of the characters of a string



Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform

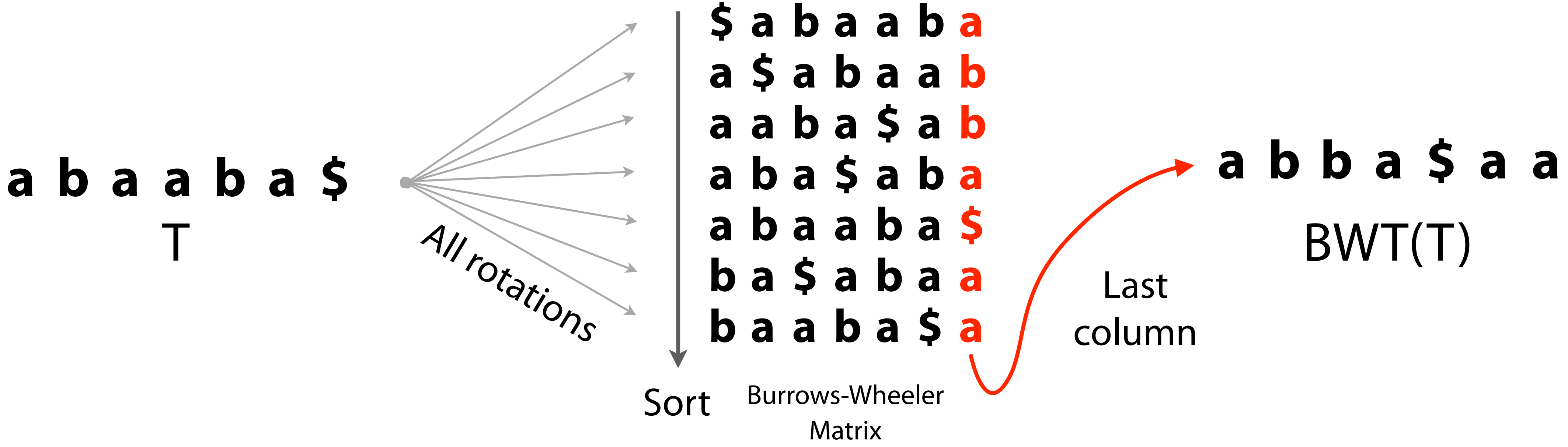
Reversible permutation of the characters of a string



Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform

Reversible permutation of the characters of a string



Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform

(1) Build all rotations

(2) Sort all rotations

(3) Take last column

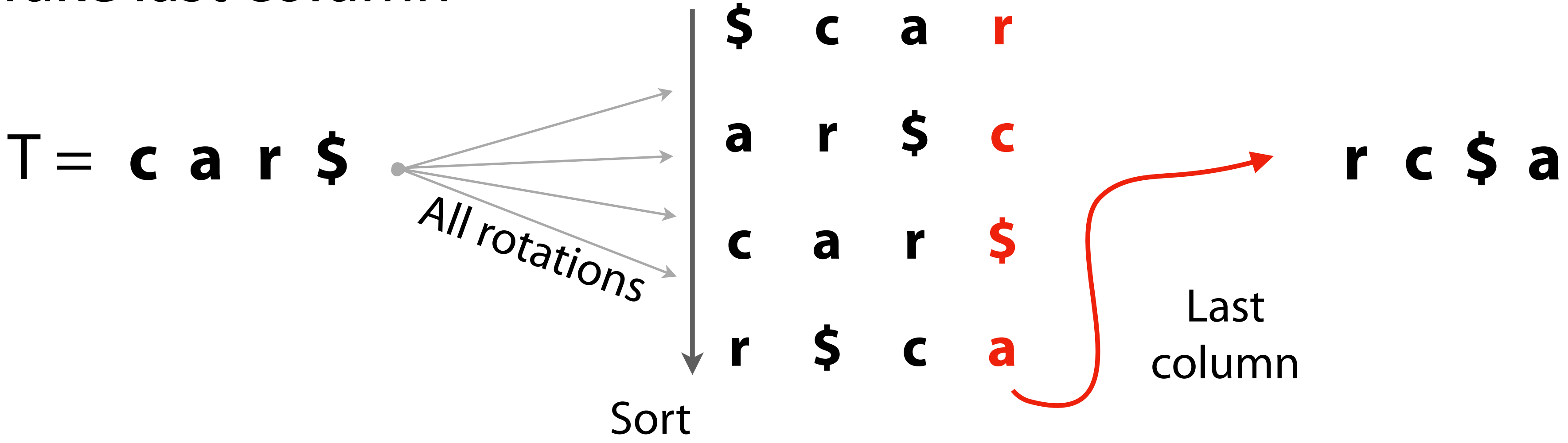
T = **c a r $**

Burrows-Wheeler Transform

(1) Build all rotations

(2) Sort all rotations

(3) Take last column



Assignment 8: a_bwt

Learning Objective:

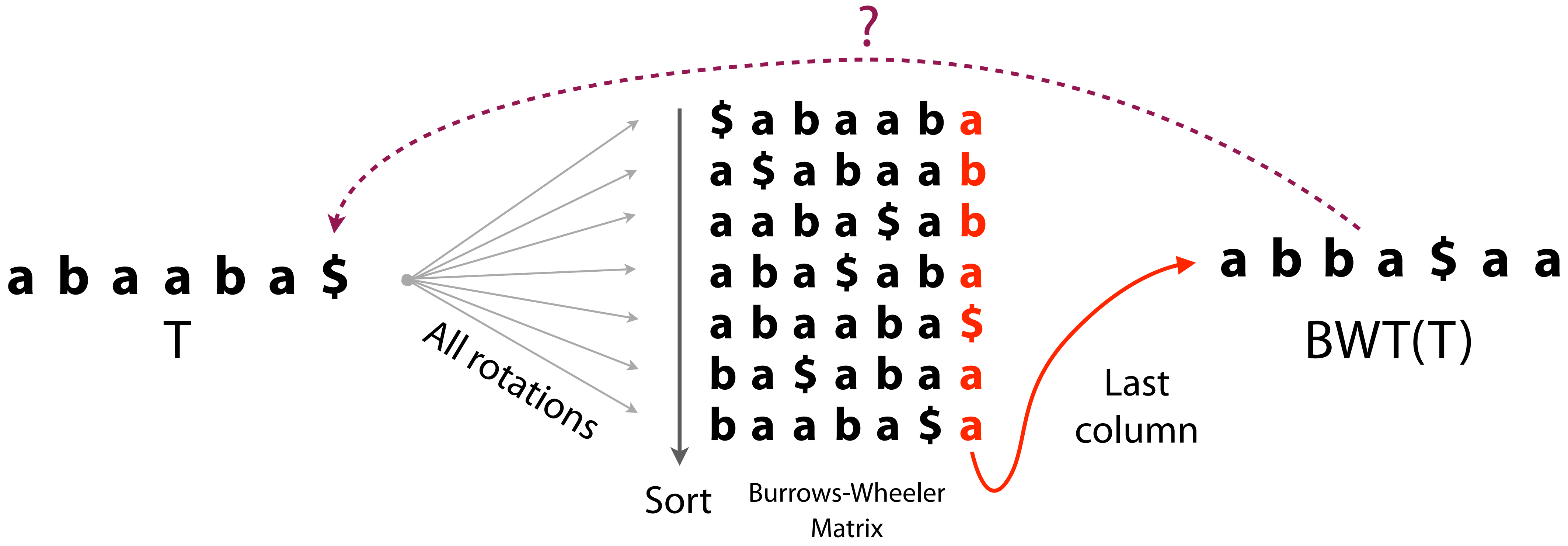
Implement the Burrows-Wheeler Transform on text

Reverse the Burrows-Wheeler Transform to reproduce text

Consider: How can the BWT be stored *smaller* than the original text?

Burrows-Wheeler Transform

How to reverse the BWT?



Burrows-Wheeler Transform

BWT(T) = **r c $ a**   T = **c a r $**

Burrows-Wheeler Transform

BWT(T) = **r c $ a**   T = **c a r $**

1) Prepend the BWT as a column

2) Sort the full matrix as rows

3) Repeat 1 and 2 until full matrix

4) Pick the row ending in ‘$’

$
a
c
r

$	**c**
a	**r**
c	**a**
r	**$**

$	**c**	**a**
a	**r**	**$**
c	**a**	**r**
r	**$**	**c**

$	**c**	**a**	**r**
a	**r**	**$**	**c**
c	**a**	**r**	**$**
r	**$**	**c**	**a**

Burrows-Wheeler Transform

BWT(T) = **r c $ a** T = **c a r $**

$	c	a	**r**	**$**
a	r	$	**c**	**a**
c	a	r	**$**	**c**
r	$	c	**a**	**r**

Burrows-Wheeler Transform

BWT(T) = **r c $ a** T = **c a r $**

$	c	a	**r**	**$**	**c**
a	r	$	**c**	**a**	**r**
c	a	r	**$**	**c**	**a**
r	$	c	**a**	**r**	**$**

Burrows-Wheeler Transform

BWT(T) = **r c $ a** T = **c a r $**

$	c	a	**r**
a	r	$	**c**
c	a	r	**$**
r	$	c	**a**

$	**c**	**a**
a	**r**	**$**
c	**a**	**r**
r	**$**	**c**

Burrows-Wheeler Transform

What is the right context of **a p p l e $** ? **l e $ a p**

A letter always has the same right context.

$	**a**	**p**	**p**	**l**	**e**
a	**p**	**p**	**l**	**e**	**$**
e	**$**	**a**	**p**	**p**	**l**
l	**e**	**$**	**a**	**p**	**p**
p	**l**	**e**	**$**	**a**	**p**
p	**p**	**l**	**e**	**$**	**a**

Burrows-Wheeler Transform: T-ranking

To continue, we have to be able to uniquely identify each character in our text.

Give each character in *T* a rank, equal to # times the character occurred previously in *T*. Call this the *T-ranking*.

a b a a b a $

Ranks aren't explicitly stored; they are just for illustration

Burrows-Wheeler Transform

BWM with T-ranking:

F						*L*
$	a_0	b_0	a_1	a_2	b_1	$\mathbf{a}_3$
$\mathbf{a}_3$	$	a_0	b_0	a_1	a_2	b_1
$\mathbf{a}_1$	a_2	b_1	a_3	$	a_0	b_0
$\mathbf{a}_2$	b_1	a_3	$	a_0	b_0	$\mathbf{a}_1$
$\mathbf{a}_0$	b_0	a_1	a_2	b_1	a_3	$
b_1	a_3	$	a_0	b_0	a_1	$\mathbf{a}_2$
b_0	a_1	a_2	b_1	a_3	$	$\mathbf{a}_0$

Look at first and last columns, called *F* and *L* (and look at just the **a**s)

as occur in the same order in *F* and *L*. As we look down columns, in both cases we see: $\mathbf{a}_3$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_0$

Burrows-Wheeler Transform

BWM with T-ranking:

F						*L*
$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	$	a_0	b_0	a_1	a_2	**b_1**
a_1	a_2	b_1	a_3	$	a_0	**b_0**
a_2	b_1	a_3	$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	$
b_1	a_3	$	a_0	b_0	a_1	a_2
b_0	a_1	a_2	b_1	a_3	$	a_0

Same with **b**s: **b_1**, **b_0**

Burrows-Wheeler Transform: LF Mapping

BWM with T-ranking:

F						*L*
$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	$	a_0	b_0	a_1	a_2	b_1
a_1	a_2	b_1	a_3	$	a_0	b_0
a_2	b_1	a_3	$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	$
b_1	a_3	$	a_0	b_0	a_1	a_2
b_0	a_1	a_2	b_1	a_3	$	a_0

LF Mapping: The i^{th} occurrence of a character *c* in *L* and the i^{th} occurrence of *c* in *F* correspond to the *same* occurrence in *T* (i.e. have same rank)

Burrows-Wheeler Transform: LF Mapping

Why does this work?

Right context:
a b a $ a b

$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a

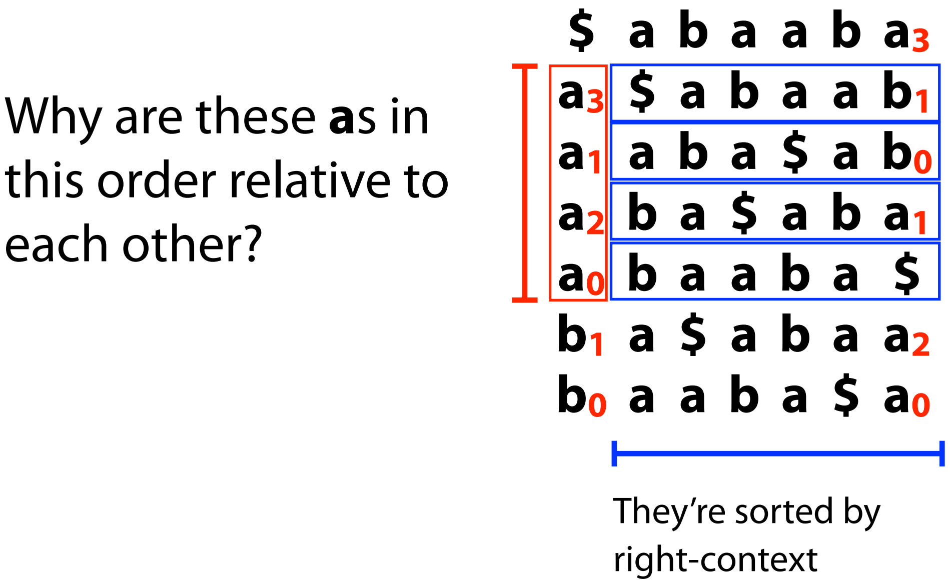
Right context:
a b a $ a b

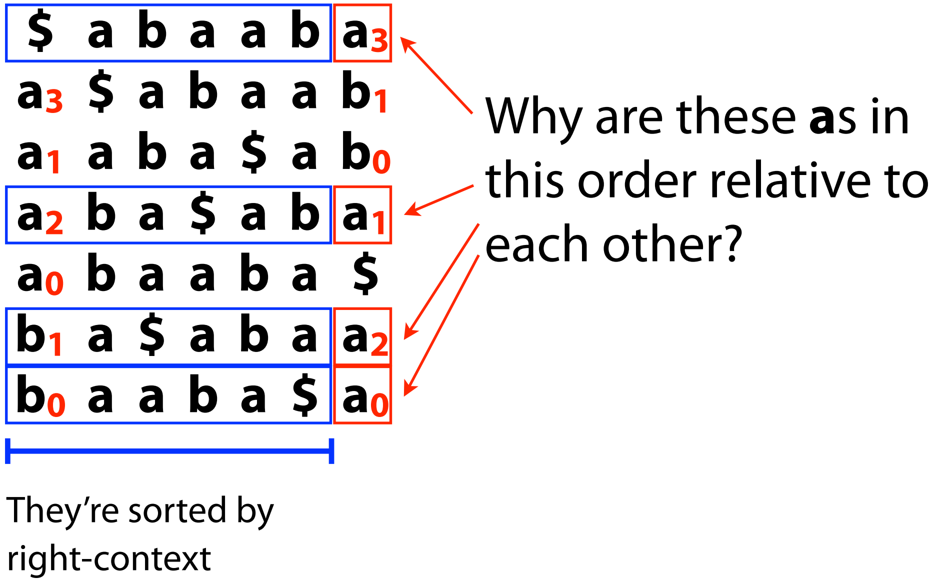
These characters have the same right contexts!

These characters *are the same character!* a_0 b_0 a_1 a_2 b_1 a_3 $

Burrows-Wheeler Transform: LF Mapping

Why does this work?





Occurrences of *c* in *F* are sorted by right-context. Same for *L*!

Any ranking we give to characters in *T* will match in *F* and *L*

Burrows-Wheeler Transform: LF Mapping

LF Mapping can be used to recover our original text too!

Given BWT = $\mathbf{a}_3\ \mathbf{b}_1\ \mathbf{b}_0\ \mathbf{a}_1\ \$\ \mathbf{a}_2\ \mathbf{a}_0$

What is L?

What is F?

Burrows-Wheeler Transform: LF Mapping

LF Mapping can be used to recover our original text too!

Start in first row. *F* must have **$**.
L contains character just prior to **$**: $\mathbf{a}_3$

Jump to row *beginning* with $\mathbf{a}_0$.
L contains character just prior to $\mathbf{a}_0$: $\mathbf{b}_0$.

Repeat for $\mathbf{b}_0$, get $\mathbf{a}_2$

Repeat for $\mathbf{a}_2$, get $\mathbf{a}_1$

Repeat for $\mathbf{a}_1$, get $\mathbf{b}_1$

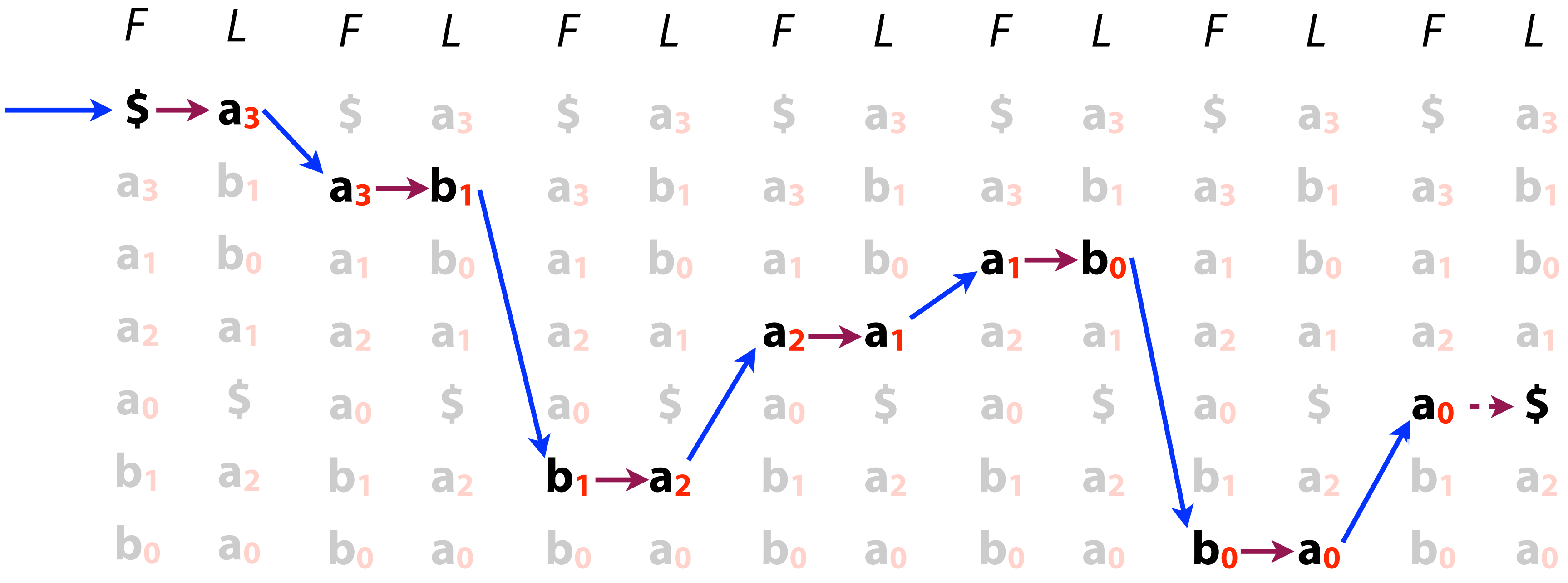
Repeat for $\mathbf{b}_1$, get $\mathbf{a}_3$

Repeat for $\mathbf{a}_3$, get **$** (done)

F	*L*
$	$\mathbf{a}_3$
$\mathbf{a}_3$	$\mathbf{b}_1$
$\mathbf{a}_1$	$\mathbf{b}_0$
$\mathbf{a}_2$	$\mathbf{a}_1$
$\mathbf{a}_0$	**$**
$\mathbf{b}_1$	$\mathbf{a}_2$
$\mathbf{b}_0$	$\mathbf{a}_0$

Burrows-Wheeler Transform: LF Mapping

Another way to visualize:



T: a_0 b_0 a_1 a_2 b_1 a_3 $

Assignment 8: a_bwt

Learning Objective:

Implement the Burrows-Wheeler Transform on text

Reverse the Burrows-Wheeler Transform to reproduce text

Consider: You can use either LF mapping or prepend-sort to reverse. Which do you think would be easier to implement (or more efficient)?

Burrows-Wheeler Transform: A better ranking

Any ranking we give to characters in *T* will match in *F* and *L*

T-Rank: Order by T

F	*L*
$	a_3
a_3	b_1
a_1	b_0
a_2	a_1
a_0	$
b_1	a_2
b_0	a_0

F-Rank: Order by F

F	*L*
$	a_0
a_0	b_0
a_1	b_1
a_2	a_1
a_3	$
b_1	a_2
b_0	a_3

What is good about f-rank?

Burrows-Wheeler Transform: A better ranking

T = **a b b c c d $**

What is the BWM index for my first instance of C? **(C_0)** [0-base for answer]

F						*L*
$	a	b	b	c	c	**d**
a	b	b	c	c	d	**$**
b	b	c	c	d	$	**a**
b	c	c	d	$	a	**b**
c	c	d	$	a	b	**b**
c	d	$	a	b	b	**c**
d	$	a	b	b	c	**c**

Burrows-Wheeler Transform: A better ranking

Say *T* has 300 **A**s, 400 **C**s, 250 **G**s and 700 **T**s and **$** < **A** < **C** < **G** < **T**

What is the BWM index for my 100th instance of G? (**G_{99}**) [0-base for answer]

Skip row starting with **$** (1 row)
Skip rows starting with **A** (300 rows)
Skip rows starting with **C** (400 rows)
Skip first 99 rows starting with **G** (99 rows)

Answer: skip 800 rows -> **index 800 contains my 100th G**

With a little preprocessing we can find any character in O(1) time!

FM Index

(Next week's material)

An index combining the BWT *with a few small auxiliary data structures*

Core of index is ***first (F)*** and ***last (L) rows*** from BWM:

F						*L*
$	a	b	a	a	b	**a**
a	$	a	b	a	a	**b**
a	a	b	a	$	a	**b**
a	b	a	$	a	b	**a**
a	b	a	a	b	a	**$**
b	a	$	a	b	a	**a**
b	a	a	b	a	$	**a**

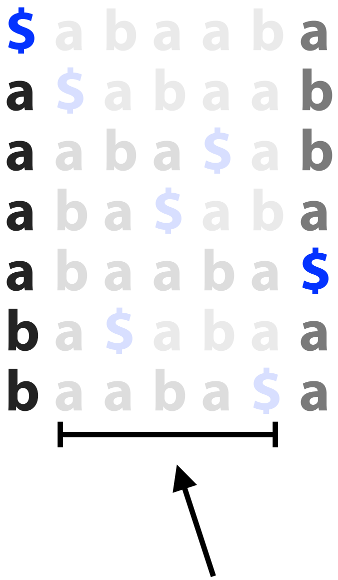
L is the same size as *T*

F can be represented as array of $|\Sigma|$ integers (or not stored at all!)

We're discarding *T* — *we can recover it from L!*

FM Index: Querying

Can we query like the suffix array?



6	$
5	a $
2	a a b a $
3	a b a $
0	a b a a b a $
4	b a $
1	b a a b a $

We don't have these columns, and we don't have T.
Binary search not possible.

FM Index: Querying

The BWM is a lot like the suffix array — maybe we can query the same way?

```
$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a
```

BWM(T)

```
6 $
5 a $
2 a a b a $
3 a b a $
0 a b a a b a $
4 b a $
1 b a a b a $
```

SA(T)

FM Index: Querying

The BWM is a lot like the suffix array — maybe we can query the same way?

\$	a	b	a	a	b	a
a	\$	a	b	a	a	b
a	a	b	a	\$	a	b
a	b	a	\$	a	b	a
a	b	a	a	b	a	\$
b	a	\$	a	b	a	a
b	a	a	b	a	\$	a

6	\$
5	a \$
2	a a b a \$
3	a b a \$
0	a b a a b a \$
4	b a \$
1	b a a b a \$

We don't have these columns, and we don't have T.

FM Index: Querying

Look for range of rows of BWM(T) with *P* as prefix

Start with shortest suffix, then match successively longer suffixes

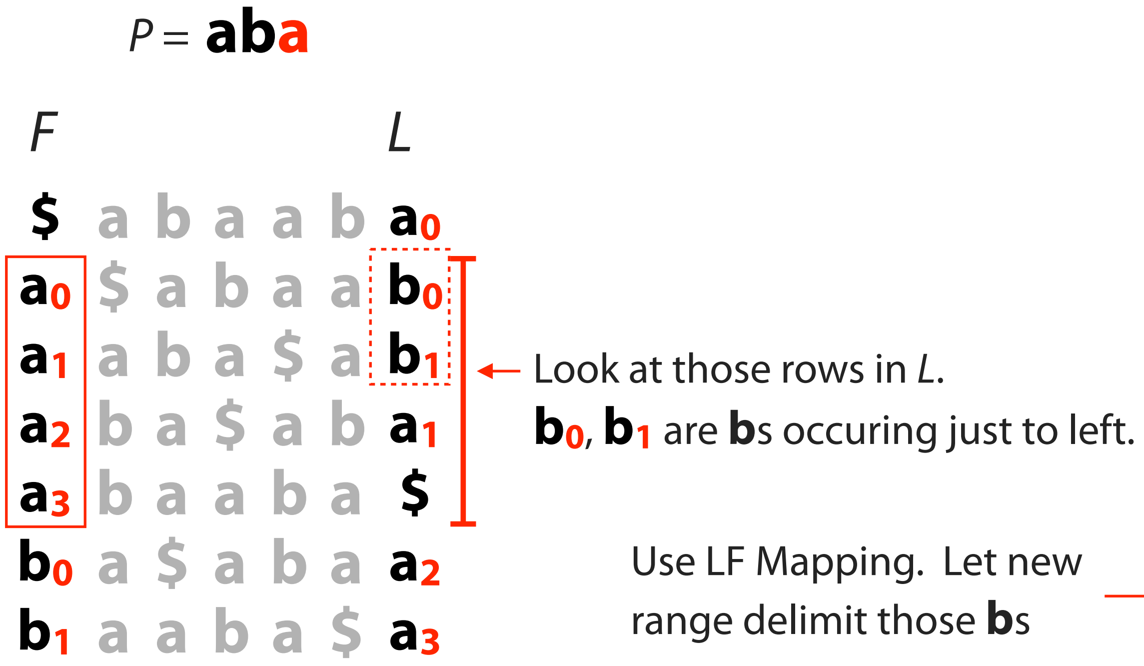
P = **aba**

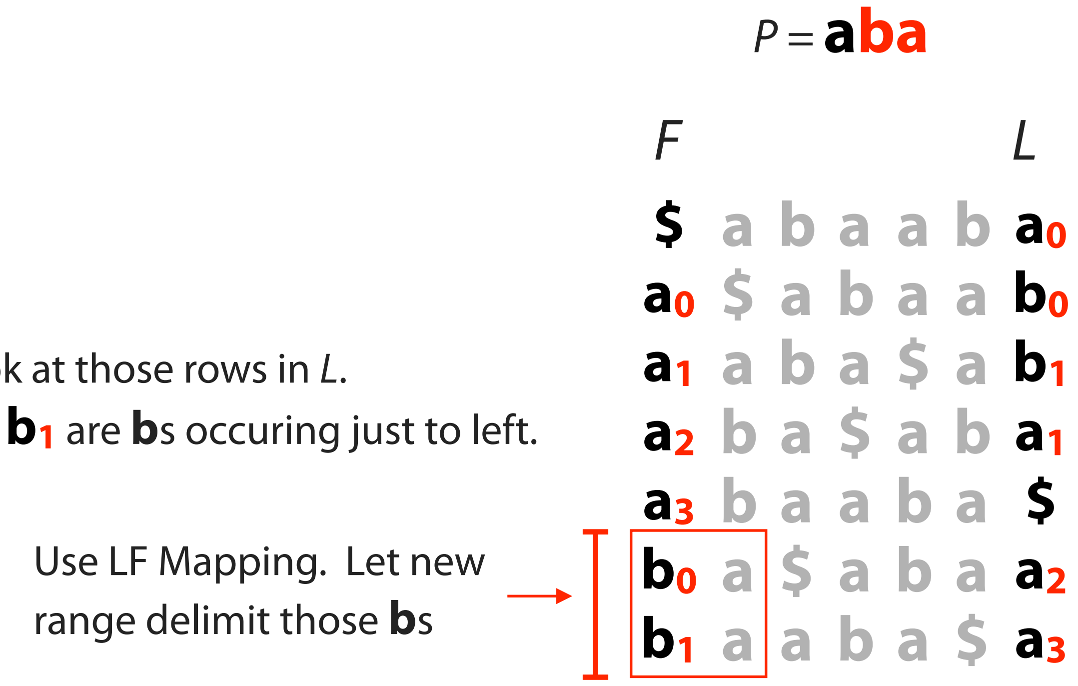
Easy to find all the rows beginning with **a**

F						L
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b
a_1	a	b	a	$	a	b
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b	a	$	a	b	a	a_2
b	a	a	b	a	$	a_3

FM Index: Querying

We have rows beginning with **a**, now we want rows beginning with **ba**

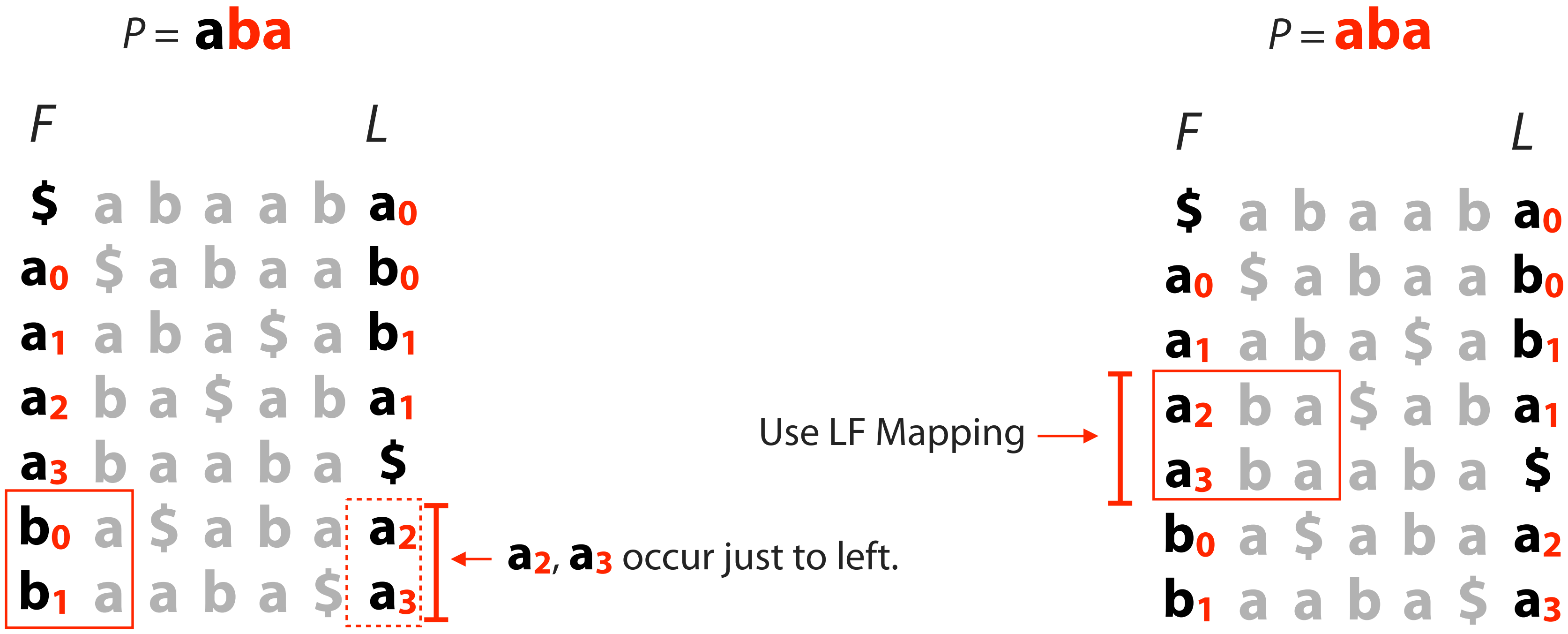




Note: We still aren't storing the characters in grey, we just know they exist.

FM Index: Querying

We have rows beginning with **ba**, now we seek rows beginning with **aba**



Now we have the rows with prefix **aba**

FM Index: Querying

When *P* does not occur in *T*, we eventually fail to find next character in *L*:

P = **bba**

F						*L*
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

Rows with **ba** prefix (b_0, b_1)

← No **b**s! (a_2, a_3)

FM Index: Querying

Problem 1: If we *scan* characters in the last column, that can be slow, $O(m)$

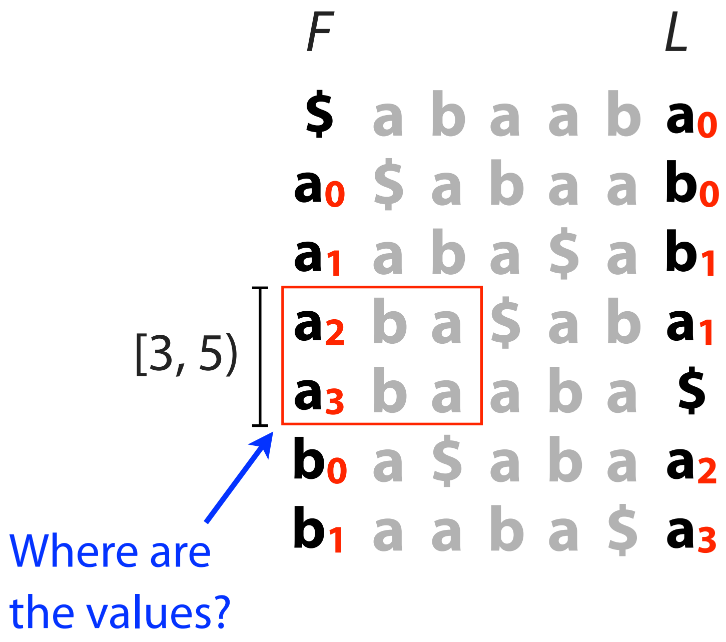
P = **aba**

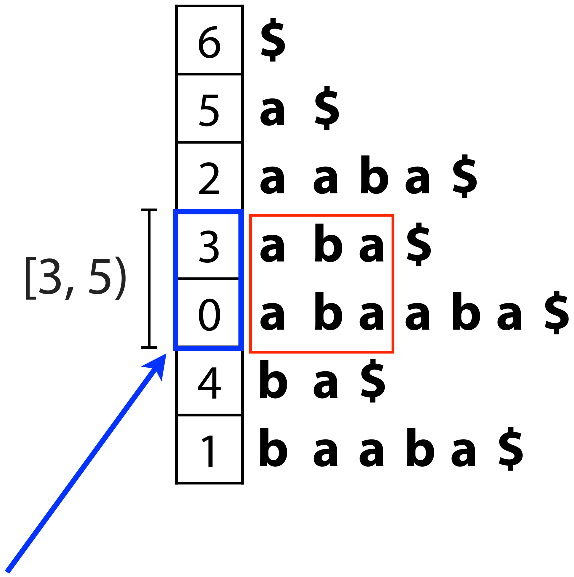
FM Index: Querying

Problem 2: We don't immediately know *where* the matches are in T...

$P =$ **aba**

Got the same range, [3, 5), we would have got from suffix array





Bonus Slides

Burrows-Wheeler Transform

Reversible permutation of the characters of a string

T		BWT(T)
B A N A N A $	⟷	A N N B $ A A

1) How to encode?

2) How to decode?

3) How is it useful for compression?

4) How is it useful for search?

Burrows-Wheeler Transform

Tomorrow_and_tomorrow_and_tomorrow

w$wwdd__nnoooaattTmmmrrrrrooo__ooo

It_was_the_best_of_times_it_was_the_worst_of_times$

s$esttssfftteww_hhmmbootttt_ii__woeeaaressIi_______

"bzip": compression w/ a BWT to better organize text

Burrows-Wheeler Transform

```
orrow_and_tomorrow_and_tomorrow$tom
ow$tomorrow_and_tomorrow_and_tomorr
ow_and_tomorrow$tomorrow_and_tomorr
ow_and_tomorrow_and_tomorrow$tomorr
row$tomorrow_and_tomorrow_and_tomor
row_and_tomorrow$tomorrow_and_tomor
row_and_tomorrow_and_tomorrow$tomor
rrow$tomorrow_and_tomorrow_and_tomo
```

Ordered by the ***context*** to the ***right*** of each character

Burrows-Wheeler Transform

In English (and most languages), the next character in a word is not independent of the previous.

In general, if text structured BWT(T) more compressible

final char (L)	sorted rotations
a	n to decompress. It achieves compression
o	n to perform only comparisons to a depth
o	n transformation} This section describes
o	n transformation} We use the example and
o	n treats the right-hand side as the most
a	n tree for each 16 kbyte input block, enc
a	n tree in the output stream, then encodes
i	n turn, set $L[i]$ to be the
i	n turn, set $R[i]$ to the
o	n unusual data. Like the algorithm of Man
a	n use a single set of probabilities table
e	n using the positions of the suffixes in
i	n value at a given point in the vector $R
e	n we present modifications that improve t
e	n when the block size is quite large. Ho
i	n which codes that have not been seen in
i	n with ch appear in the {\em same order
i	n with ch. In our exam
o	n with Huffman or arithmetic coding. Bri
o	n with figures given by Bell~\cite{bell}.

Figure 1: Example of sorted rotations. Twenty consecutive rotations from the sorted list of rotations of a version of this paper are shown, together with the final character of each rotation.

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform

Lets compare the SA with the BWT…

T = **a b a a b a $**

6
5
2
3
0
4
1

SA(T)

$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a

BWM(T)

Suffix Array is O(m)

Burrows-Wheeler Transform

Lets compare the SA with the BWT…

T = **a b a a b a $**

6
5
2
3
0
4
1

SA(T)

a
b
b
a
$
a
a

BWT(T)

Suffix Array is O(m)

BWT is O(m)

The BWT has a better constant factor!

Burrows-Wheeler Transform

BWM is related to the suffix array

```
$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a
```

BWM(T)

```
6  $
5  a $
2  a a b a $
3  a b a $
0  a b a a b a $
4  b a $
1  b a a b a $
```

SA(T)

Same order whether rows are rotations or suffixes

Burrows-Wheeler Transform

In fact, this gives us a new definition / way to construct BWT(T):

$$BWT[i] = \begin{cases} T[SA[i]-1] & \text{if } SA[i] > 0 \\ \$ & \text{if } SA[i] = 0 \end{cases}$$

“BWT = characters just to the left of the suffixes in the suffix array”

a b a a b a $

T

6	**$**
5	**a $**
2	**a a b a $**
3	**a b a $**
0	**a b a a b a $**
4	**b a $**
1	**b a a b a $**

SA(T)

BWM(T)

Burrows-Wheeler Transform

In fact, this gives us a new definition / way to construct BWT(T):

$$BWT[i] = \begin{cases} T[SA[i]-1] & \text{if } SA[i] > 0 \\ \$ & \text{if } SA[i] = 0 \end{cases}$$

"BWT = characters just to the left of the suffixes in the suffix array"