Q1: Inductive step for inequality proof

Question prompt: The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$.

For any positive integer k, prove that if $\prod_{p=1}^{k} \frac{2p-1}{2p} < \frac{1}{\sqrt{2k+1}}$, then $\prod_{p=1}^{k+1} \frac{2p-1}{2p} < \frac{1}{\sqrt{2k+3}}$.

Hints: Work backwards from the goal, then rewrite into logical order. Try squaring both sides. For positive numbers a and b, $a < b$ if and only if $\sqrt{a} < \sqrt{b}$.

Solution:

Let k be a positive integer and suppose that $\prod_{p=1}^{k} \frac{2p-1}{2p} < \frac{1}{\sqrt{2k+1}}$. Notice that $(2k+1)(2k+3) = 4k^2+8k+3 < 4k^2+8k+4 = (2k+2)^2$. So $\frac{2k+1}{(2k+2)^2} < \frac{1}{2k+3}$. So $\frac{(2k+1)^2}{(2k+2)^2} < \frac{2k+1}{2k+3}$. Taking the square root of both sides gives us $\frac{2k+1}{2k+2} < \frac{\sqrt{2k+1}}{\sqrt{2k+3}}$. And therefore $\frac{2k+1}{2k+2}\frac{1}{\sqrt{2k+1}} < \frac{1}{\sqrt{2k+3}}$. Using this fact and the given fact that $\prod_{p=1}^{k} \frac{2p-1}{2p} < \frac{1}{\sqrt{2k+1}}$, we have $\prod_{p=1}^{k+1} \frac{2p-1}{2p} = \frac{2k+1}{2k+2}(\prod_{p=1}^{k} \frac{2p-1}{2p}) < \frac{2k+1}{2k+2}\frac{1}{\sqrt{2k+1}} < \frac{1}{\sqrt{2k+3}}$ So $\prod_{p=1}^{k+1} \frac{2p-1}{2p} < \frac{1}{\sqrt{2k+3}}$, which is what we needed to show.

Q2: Set inclusion proof

Question prompt:

$A = \{(a, b) \in R^2 \; : \; |a + b| \leq 2\}$
$B = \{(x, y) \in R^2 \; : \; |x - y + 7| \leq 1\}$
$C = \{(p, q) \in R^2 \; : \; p \leq 0 and q \geq 0\}$
Prove that $A \cap B \subseteq C$.

Solution:

Let $(x, y) \in R^2$ and suppose that $(x, y) \in A \cap B$. Then $(x, y) \in A$ and $(x, y) \in B$.

Since $(x, y) \in A$, $|x + y| \leq 2$. Since $(x, y) \in B$, $|x - y + 7| \leq 1$.

So $x + y \leq 2$ and $x - y + 7 \leq 1$. Adding these equations together, we get $2x + 7 \leq 3$. So $2x \leq -4$. And therefore $x \leq -2 \leq 0$.

Since $|x + y| \leq 2$, it's also the case that $-x - y \leq 2$. Adding this to $x - y + 7 \leq 1$, we get $-2y + 7 \leq 3$. So then $-2y \leq -4$. So $y \geq 2 \geq 0$.

So $x \leq 0$ and $y \geq 0$, so $(x, y) \in C$, which is what we needed to prove.

Q3: One-to-one proof

Note that onto proofs may appear on the final as well.

Question prompt:

Suppose that $f : [0, \frac{1}{2}] \to [1, \frac{5}{2}]$ is defined by $f(x) = \frac{x^2+1}{1-2x^2}$ Prove that f is one-to-one.

You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

Solution:

Let x and y be any numbers in $[0, \frac{1}{2}]$ and suppose $f(x) = f(y)$, that is

$$\begin{aligned}
& \frac{x^2+1}{1-2x^2} = \frac{y^2+1}{1-2y^2} \\
\Rightarrow\quad & (x^2+1)(1-2y^2) = (y^2+1)(1-2x^2) \\
\Rightarrow\quad & x^2+1-2x^2y^2-2y^2 = y^2+1-2x^2y^2-2x^2 \\
\Rightarrow\quad & 3x^2 = 3y^2 \\
\Rightarrow\quad & x = y
\end{aligned}$$

(The last step works because x and y are both positive.)
Therefore f is one-to-one.

Q4 Tree induction

Question prompt:

Recall that a node in a full binary tree is either a leaf or has exactly two children. A Merry tree is a full binary tree whose nodes contain integers, such that the root node contains a positive integer and all leaf nodes contain 0.

Use (strong) induction to prove that a Merry tree contains a node whose value is larger than the values of both its children.

Solution:

Induction variable: The induction variable is h, which is the height of the tree.

Base Case(s): The shortest Merry trees have height $h = 1$. The root contains a positive number and its children (i.e. the leaves) contain 0. So the root node has a value larger than the values of both its children.

Inductive Hypothesis: Suppose that every Merry tree a node whose value is larger than the values of both its children, for heights $h = 1, \ldots, k-1$.

Inductive Step: Consider a Merry tree T with height k. T consists of a root node N containing value x and two child subtrees T_L and T_R. Suppose that the values in the roots of T_L and T_R are y and z. Note that T_L and T_R are not necessarily Merry trees.

There are three cases:

Case 1: $x > y$ and $x > z$. Then N is the required node with a value larger than both its children.

Case 2: $x \leq y$. Since we know that x is positive, then y must be positive. So T_L is a Merry tree. Therefore, by the inductive hypothesis, T_L contains a node with a value larger than both its children.

Case 3: $x \leq z$. Since we know that x is positive, then z must be positive. So T_R is a Merry tree. Therefore, by the inductive hypothesis, T_R contains a node with a value larger than both its children.

In all three cases, T contains a node with a value larger than both its children, which is what we needed to prove.

Q5: Inequality induction on recursion

Question prompt:

Suppose that $f : Z^+ \to Z^+$ is defined by:
$f(1) = 3$
$f(2) = 7$
$f(n) = f(n-1) + 2f(n-2)$, for all $n \geq 3$
Use (strong) induction to prove that $f(n) \leq 3^n$

Solution:

Proof by induction on n.
Base case(s): For $n = 1$, $f(n) = 3$ and $3^n = 3$, so $f(n) \leq 3^n$.
For $n = 2$, $f(n) = 7$ and $3^n = 3^2 = 9$, so $f(n) \leq 3^n$.
So the claim holds.
Inductive hypothesis [Be specific, don't just refer to "the claim"]:
Suppose that $f(n) \leq 3^n$, for $n = 1, 2, \ldots, k-1$, for some integer $k \geq 3$.
Rest of the inductive step:
By the inductive hypothesis, we know that $f(k-1) \leq 3^{k-1}$ and $f(k-2) \leq 3^{k-2}$. So, using these two inequalities plus the definition of f, we get:
$f(k) = f(k-1) + 2f(k-2) \leq 3^{k-1} + 2 \cdot 3^{k-2}$
But then $3^{k-1} + 2 \cdot 3^{k-2} \leq 3^{k-1} + 2 \cdot 3^{k-1} = 3 \cdot 3^{k-1} = 3^k$
So $f(k) \leq 3^k$, which is what we needed to show.

Q6: Combinations

Note that permutation problems may appear on the final as well.

Question prompt:

A domino has two ends, each of which may be blank or contain between one and n spots. The two ends may have the same number of spots or different numbers of spots. A double-n domino set contains exactly one of each possible dot combination, where the order of the two ends doesn't matter. For example, a double-two domino set contains $(0,0)$, $(1,0)$, $(2,0)$, $(1,1)$, $(1,2)$, and $(2,2)$. Give a general formula for the number of dominoes in a double-n set.

Briefly justify your answer and/or show work.

Solution:

The double-n set contains $n+1$ dominos with the same number of spots on both ends (e.g. (0, 0), (1, 1),... (n, n)).

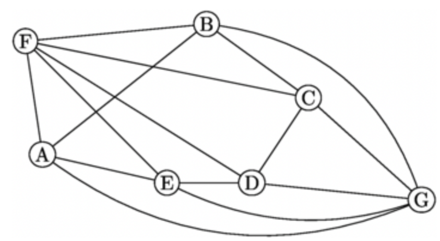
A domino with dissimilar ends corresponds to a set of two numbers in the range 0 through n. There are $\binom{n+1}{2} = \frac{n(n+1)}{2}$ such sets.

So, in total, there are $(n+1) + \frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}$ dominoes in the set.

Q7: Chromatic number

Question prompt:

What is the chromatic number of the graph below? Justify your answer.



Solution:

The chromatic number is four. To show that four is an upper bound, we can color it as follows: A, D are red, C, E are blue, F, G is green, B is yellow.

For the lower bound, the graph contains a W_5 whose hub is F and whose rim contains nodes A, B, C, D, E. Coloring a W_5 requires four colors.

Q8: Grammars

Question prompt:

Consider the following grammar G, with start symbol S and terminals a and b.

$$S \rightarrow a\ S\ b \mid b\ S\ a \mid a \mid b$$

Chandra claims that this generates the set of strings containing a's and/or b's whose length is odd. Is this correct? Justify your answer.

(We say that a grammar generates a set of strings iff it can generate **every single string** in the set and also can **not** generate any strings outside that set.)

Solution:

The wording of this problem is a somewhat ambiguous. One may interpret it as "the total length of the string is odd", whereas another interpretation could be "the number of a's is odd, and the number of b's is odd". On the actual final examlet, there won't be problems with such ambiguous wording.

Interpretation 1: Because each time this grammar makes a recursive call (using aSb or bSa) it increases the total length by 2, and the base cases (a and b) have length 1, the total length should be $2n + 1$ where n is the number of recursive calls, suggesting that the length is always odd. Therefore, the claim is true.

Interpretation 2: This grammar cannot generate the string aaabaaa, whose number of a's and number of b's are both odd. Based on the grammar, any string with length greater than 1 must have different head and tail (i.e. if it starts with a, it must end with b, and vice versa). Therefore, this grammar cannot generate all the strings whose a's and b's are odd.

Q9: Big-O

Question prompt:

Suppose that f, g, and h are functions from the reals to the reals, such that $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$. Must $f(x)g(x)$ be $O(h(x)h(x))$? Briefly justify your answer.

Solution:

This is true. Since $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$, there are positive reals c, k, C and K such that $0 \leq f(x) \leq ch(x)$ and $0 \leq g(y) \leq Ch(y)$ for every $x \geq k$ and $y \geq K$.

But then if we let $p = cC$, we have $0 \leq f(x)g(x) \leq ph(x)h(x)$ for every $x \geq \max(k, K)$, which suggests that $f(x)g(x)$ is $O(h(x)h(x))$.

Q10: Proof by contradiction

Proof by contradiction problems will not appear on the final examlet. You are safe and free to skip this one.

Question prompt:

Use proof by contradiction to show that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.

Solution: Suppose not. That is, suppose that there are positive integers x and y such that $x^2 - y^2 = 10$. Factoring the lefthand side, we get $(x - y)(x + y) = 10$. $(x - y)$ and $(x + y)$ must be integers since x and y are integers.

Ignoring sign, there are only two ways to factor 10: $2 \cdot 5$ or $1 \cdot 10$. In both cases, exactly one of the factors is odd, so the sum of the two factors is odd.

But the sum of $(x - y)$ and $(x + y)$ is $2x$, which is even.

We have found a contradiction, so the original claim must have been correct.

Q11: Unrolling

Question prompt:

Suppose we have a function g defined (for n a power of 2) by ¡/p¿

$$
\begin{aligned}
g(1) &= 1 \\
g(n) &= 4g(n/2) + n^2 forn \geq 2
\end{aligned}
$$

Your partner has already figured out that

$$g(n) = 4^k g(n/2^k) + kn^2$$

Finish finding the closed form for g. Show your work and simplify your answer (in particular, $\log(n)$ should never appear as an exponent; you may find this identity useful: $a^{\log_b n} = n^{\log_b a}$)

Solution:

To find the value of k at the base case, set $n/2^k = 1$. Then $n = 2^k$, so $k = \log_2 n$. Substituting this into the above, we get:

$$
\begin{aligned}
g(n) &= 4^k g(n/2^k) + kn^2 \\
&= 4^{\log_2 n} g(1) + (\log_2 n)n^2 \\
&= 4^{\log_2 n} + n^2 \log_2 n \\
&= n^{\log_2 4} + n^2 \log_2 n \\
&= n^2 + n^2 \log_2 n
\end{aligned}
$$

Q12: Recursion tree

Question prompt:

Suppose that T is defined as follows, for even integer n.

$$\begin{aligned} T(0) &= 5 \\ T(n) &= 3T(n-2) + n^2 \end{aligned}$$

We fix k such that level k in T's recursion tree is an internal (i.e. non-leaf) level. Calculate the total work (i.e. the sum of the node values) at level k. Show your work.

Solution:

Because each time we make 3 recursive calls, each internal node should have 3 children. Therefore, at level k we have 3^k nodes.

For each level, the input size is reduced by 2, so at level k, the input size should be $n - 2k$.

The non-recursive part of the recursive definition suggests that at each node the work is $f(x) = x^2$. Now if we plug in the input size of $n - 2k$ at level k, each node should have work $(n - 2k)^2$.

Therefore, the total work at level k is $3^k(n - 2k)^2$