Discussion Solutions Week 7

CS 173: Discrete Structures

Tuesday

Problem 18.2. in Discussion Manual

(a)
```
states:   3          4              fail
actions:    password   unauthorized
```

(b) $S = \{start, 1, 2, 3, 4, 5, done, error, finished, fail\}$

$A = \{requestpage, login, shippage, notfound, password, unauthorized\}$

(c) $\delta(1, requestpage) = \emptyset$

$\delta(3, password) = \{4\}$

$\delta(start, requestpage) = \{1, 2, 5\}$

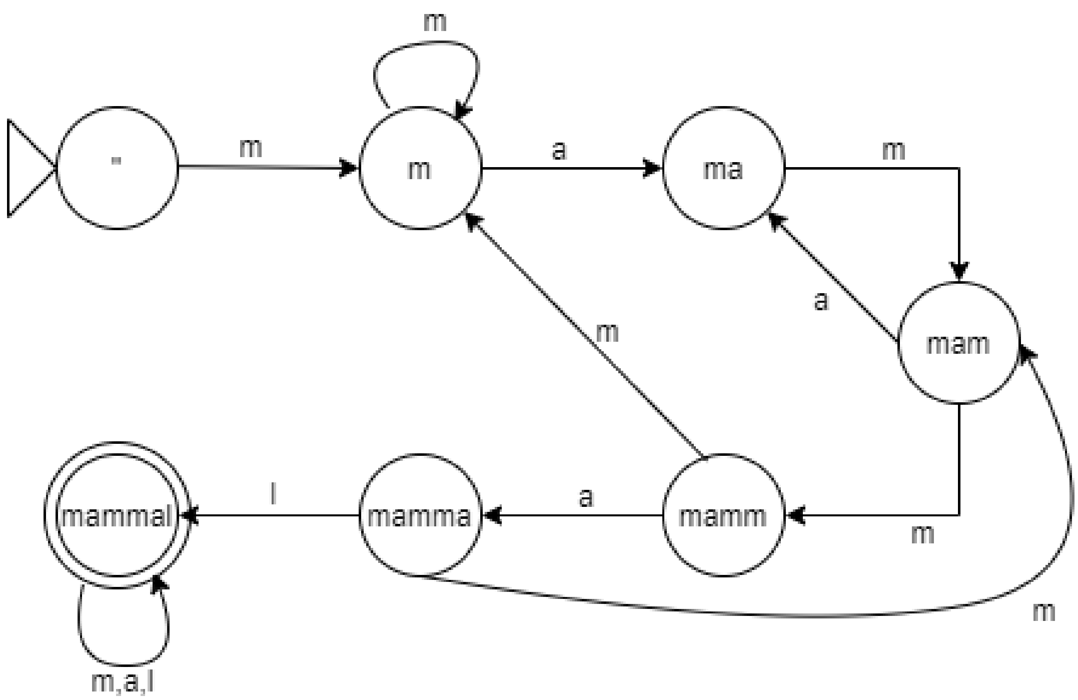
(d) $10 \cdot 6 = 60$. 7 are non-empty.

(e) $1, 2, 5$ can be combined into a single state. You could almost certainly combine "done" and "finished"; you could probably also combine "error" and "fail", though it depends on how this information is likely to be used (e.g. do we want later processing to be able to report different specific errors). *(This problem is a bit open-ended and open to interpretation.)*

Wednesday

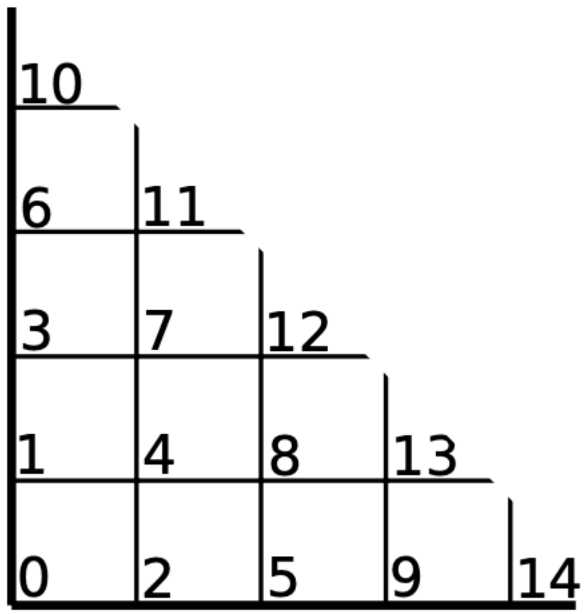
Problem Magic word. from State Diagrams

All three actions are possible from any state, but for readability, all arrows returning to the start state have been omitted from the diagram below (so e.g. there is an implicit arrow with label “l” from “mam” back to the start state).



Thursday

Problem 19.2. in Discussion Manual

(a) 

(b) Consider the values of x, y satisfying $x + y = k$.

Because we are in $\mathbb{N}$, for any such values of x and y we have that $y \geq 0$ and therefore $x \leq k$. For any value $x \leq k$, we can let $y = k - x$ to achieve $x + y = k$.

Thus, x ranges from 0 to k, and $f(x, y) = s(x + y) + x = s(k) + x$ ranges from $s(k)$ to $s(k) + k$. Remembering from lecture that $s(k) = \frac{k(k+1)}{2}$, we can also write this as:

$$\frac{k(k+1)}{2} \leq f(x, y) \leq \frac{k(k+1)}{2} + k$$

(c) The preimage of 17 is $\{(2, 3)\}$. Note that $f(2, 3) = s(5) + 2 = 15 + 2 = 17$.

We can show that $(2, 3)$ is the only element in the pre-image by noting from our solution to part d) that, for all x, y, if $f(x, y) = f(2, 3)$, then $x + y = 2 + 3 = 5$. Testing all such values of x and y shows that $(2, 3)$ is the only element in the pre-image of 17.

(Alternatively, we could argue that there can't be any other element in the pre-image because, as demonstrated through parts (d) and (e), f is one-to-one.)

(d) Let $k = x + y$, $l = p + q$. From the given supposition we know $k \neq l$, so without loss of generality, assume that $k < l$.

We get the following:

$$
\begin{aligned}
f(x,y) &\leq \frac{k(k+1)}{2} + k \qquad \text{[from part (b)]} \\
&= \frac{k^2+3k}{2} \\
&< \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&\leq \frac{l(l+1)}{2} \qquad [k < l, \text{ and } k, l \in \mathbb{Z}, \text{ so } k+1 \leq l] \\
&\leq f(p,q) \qquad \text{[from part (b)]}
\end{aligned}
$$

This establishes $f(x,y) < f(p,q)$, so $f(x,y) \neq f(p,q)$, QED.

(e) Suppose not. That is, suppose towards a proof by contradiction that $f(x,y) = f(p,q)$. Further, let $k = x + y = p + q$. Then:

$$
\begin{aligned}
f(x,y) &= f(p,q) \\
s(x+y) + x &= s(p+q) + p \\
s(k) + x &= s(k) + p \\
x &= p
\end{aligned}
$$

Since $x = p$ and $x + y = p + q$, we have that $y = q$. But we assumed that $(x,y) \neq (p,q)$, contradiction. So our initial supposition must be false, and thus instead we know $f(x,y) \neq f(p,q)$; QED.

Friday

Problem 19.1. in Discussion Manual

(a) **Countably infinite**. In fact it's basically the definition of countably infinite - the bijection mapping it to $\mathbb{N}$ is $id_{\mathbb{N}}$.

(b) **Uncountable**. The powerset of a set always has a (strictly) larger cardinality than that set. *(Or a handwavy 'solution' thinking about representations: these do not appear to all have finite representations - if I have an infinite set of naturals with no pattern, how would I possibly write down that set?)*

(c) **Uncountable**. We know $\mathbb{R}$ is uncountable, and $\mathbb{R} \subseteq \mathbb{C}$.

(d) **Countably infinite**. There are clearly infinitely many elements (in particular at the very least there are the elements $\{0\}$, $\{1\}$, $\{2\}$, $\cdots$). To show it's *countably* infinite, we can provide a one-to-one function f mapping these to the (finite) bit strings: given S with maximum element n, return the bit string of length $n+1$ with a 1 in (0-indexed) position i iff $i \in S$. For example, $f(\{0,3,4\}) = 10011$. And we know the set of bit strings (or any other strings with a finite alphabet) is countable. *(Alternatively, thinking with representations: each $S \in X$ has a roster notation which is finite - e.g.* $\{0,3,4\}$*.)*

(e) **Countably infinite**. There are clearly infinitely many elements (in particular, at the very least there are the silly books containing just "a", "aa", "aaa", $\cdots$). And it's *countably* infinitely because each book is just one (finite) string created using a fixed (finite) alphabet. *(You may be tempted to think of a book as a* list *of strings separated by spaces, but that's making it more complicated than necessary - there's no need to treat characters like space and newline any differently from a and b.)*

(f) **Countably infinite**. We know $\mathbb{Q}$ is countable, and this set is a subset of $\mathbb{Q}$. *(Thinking with representations: these are reals specifically chosen to have expansions that end - i.e. representations that are finite.)*

Problem Additional Problem. from Countability

Lemma: For sets A and B, there exists a one-to-one function $f : A \to B$ if and only if there exists an onto function $g : B \to A$.

Proof: See solution to the "additional tutorial problem" from the Functions week - the only difference is that now we are working with arbitrary sets instead of subsets of $\mathbb{N}$, so where that solution uses the function *minimum* (which can choose a representative from a set of naturals), we instead have to use the choice function h from the hint. □

We know that by definition, there exists a one-to-one function $f : A \to B$ if and only if $|A| \leq |B|$. So now by the lemma, we've established that there exists an onto function $g : B \to A$ if and only if $|A| \leq |B|$.[1]

[1] Optional extra details: The lemma we used actually does not hold if $A = \emptyset$ and $B \neq \emptyset$ (can you find the flaw in the argument?), so our new cardinality definition would have to special-case that situation by specifying that $|\emptyset| < |B|$ for all non-empty B. Can you see why $\emptyset$ does *not* have to be treated as a special case in our normal one-to-one-based definition for cardinality?