Discussion Solutions Week 4

CS 173: Discrete Structures

Tuesday

Problem 11.1. in Discussion Manual

(a) Proof by induction on n.

Base: Let $n = 1$. Then $\sum_{n=1}^{1} i^2 = 1 = \frac{1(2)(3)}{6}$.

Induction: Suppose (as our Inductive Hypothesis) that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n = 1...k-1$. Then our goal is to show $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

$$\begin{aligned}
\sum_{i=1}^{k} i^2 &= \sum_{i=1}^{k-1} i^2 + (k^2) && \text{(pulling a term out of the summation)} \\
&= \frac{(k-1)(k)(2(k-1)+1)}{6} + k^2 && \text{(by the Inductive Hypothesis)} \\
&= \frac{(k-1)(k)(2k-1)}{6} + k^2 && \text{(this and remaining steps are just algebra)} \\
&= \frac{(k-1)(k)(2k-1)}{6} + \frac{6k^2}{6} \\
&= \frac{(k)(k-1)(2k-1) + 6k^2}{6} \\
&= \frac{(2k^3 - 3k^2 + k) + 6k^2}{6} \\
&= \frac{2k^3 + 3k^2 + k}{6} \\
&= \frac{k(2k^2 + 3k + 1)}{6} \\
&= \frac{k(2k+1)(k+1)}{6}
\end{aligned}$$

(b) We proceed by induction on n.

Base: Let $n = 1$. Then $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{(1+1)}$.

Induction: Suppose (as our Inductive Hypothesis) that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for each $n \in \mathbb{Z}^+$

less than some positive integer r. Then our goal is to show $\sum_{k=1}^{r} \frac{1}{k(k+1)} = \frac{r}{r+1}$.

$$\begin{aligned}
\sum_{k=1}^{r} \frac{1}{k(k+1)} &= \sum_{k=1}^{r-1} \frac{1}{k(k+1)} + \frac{1}{r(r+1)} && \text{(pulling a term out of the summation)} \\
&= \frac{(r-1)}{(r-1)+1} + \frac{1}{r(r+1)} && \text{(by the Inductive Hypothesis)} \\
&= \frac{(r-1)}{r} + \frac{1}{r(r+1)} && \text{(this and remaining steps are just algebra)} \\
&= \frac{(r-1)(r+1)+1}{r(r+1)} \\
&= \frac{r^2}{r(r+1)} \\
&= \frac{r}{r+1}
\end{aligned}$$

Problem 11.4. in Discussion Manual

(Commentary: Obviously the proof must be wrong since the claim it is proving is clearly false. While that is not enough to say where *the flaw in the proof is, it does give us a good place to check: $P(1)$ is true but $P(2)$ is false, so we should look at the inductive step and carefully audit its argument that $P(1) \rightarrow P(2)$.)*

The argument implicitly relies on the fact that S' and S'' are not disjoint. If the sets overlap by even one horse H_*, then the proof is correct that all horses in the union are the same color, since all the horses in S' are H_*'s color and so are the horses in S''. However, consider the argument in the inductive step when $k = 2$. In this case, $S' = \{H_2\}$ and $S'' = \{H_1\}$, which are disjoint. Thus while it is true that all the horses in S' are the same color and all the horses in S'' are the same color, it is wrong for the proof to claim from this that all the horses in the union must also be the same color.

Wednesday

Problem 11.1. in Discussion Manual

(c) Proof by induction on n.

Base: Let $n = 0$. Then $(\sum_{i=0}^{0} i)^2 = 0 = \sum_{i=0}^{0} i^3$.

Induction: Fix k and suppose that $(\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3$ for $n = 0, 1, \cdots, k-1$. Then we get the following:

$$\begin{aligned}
(\sum_{i=0}^{k} i)^2 &= (\sum_{i=0}^{k-1} i + k)^2 \\
&= (\sum_{i=0}^{k-1} i)^2 + 2k(\sum_{i=0}^{k-1} i) + k^2 \\
&\overset{IH}{=} \sum_{i=0}^{k-1} i^3 + 2k(\sum_{i=0}^{k-1} i) + k^2 \\
&= \sum_{i=0}^{k-1} i^3 + 2k\frac{(k-1)k}{2} + k^2 && \text{(by the given hint)} \\
&= \sum_{i=0}^{k-1} i^3 + k^3 \\
&= \sum_{i=0}^{k} i^3
\end{aligned}$$

Problem 11.3. in Discussion Manual

Induction variable: The induction variable is a, which is the area of the bar (or the number of small squares that make up the bar).

Base Case(s): At $a = 1$, the bar already consists of a single square. So we don't need to break it up further. That is, we need 0, i.e. $a - 1$ breaks to divide it up. So the claim is true.

Inductive Hypothesis: Any chocolate bar of area a can be divided into individual squares using $a - 1$ breaks, for $a = 1 \ldots k - 1$.

Inductive Step: Suppose that B is a chocolate bar of area k, where $k > 1$. Let's break B along any grid line, creating two bars X and Y.

Let x and y be the areas of X and Y. Then $x + y = k$.

By the inductive hypothesis, we can reduce X to individual squares using $x-1$ breaks. Similarly, we can reduce Y to individual squares using $y - 1$ breaks.

Therefore, to reduce B to individual squares, we use our initial break, then break up X and Y using $x - 1$ and $y - 1$ breaks. So the total number of breaks required to divide up B is $1 + (x - 1) + (y - 1) = x + y - 1 = k - 1$. So breaking up B requires $k - 1$ breaks, which is what we needed to prove.

Problem 14.1. in Discussion Manual

(a) Proof by induction on n.

Base: When $n = 8$, we have $64 > 56 + 1$. This is true.

Inductive Hypothesis: Suppose that $n^2 > 7n + 1$ for any $n = 8...k - 1$. Now our goal is to show that $k^2 > 7k + 1$.

Inductive step: $k^2 = (k-1)^2 + 2k - 1$. By the IH, $(k-1)^2 + 2k - 1 > 7(k-1) + 1 + 2k - 1 = 7k - 7 + 1 + 2k - 1 = 9k - 7$. Since $k > 8$, $9k - 7 > 7k + 1$. Thus, $k^2 > 7k + 1$.

Thursday

Problem 1. from Invalid Recursion

- f is valid.
- g is invalid because neither case covers $n = 7$. (This also means that g is not defined for any larger value of n, e.g. $g(8)$ is undefined because its definition relies on $g(7)$.)
- h is invalid because $h(7)$ and above are undefined: $h(7)$ is defined in terms of $h(8)$, which is defined in terms of $h(9)$, etc in an *infinite* chain that never reaches a base case. (You could attempt to resolve this by reading this definition 'in reverse', i.e. if $h(n) = n + h(n+1)$ then $h(n+1) = h(n) - n$, which looks more like a valid definition. But notice that there is still no way to compute $h(7)$: you can't say $h(7) = h(6+1) = h(6) - 6$ because when $n = 6$ the "$n + h(n+1)$" case of h's definition does not apply.)
- s is invalid because both cases include $n = 7$ yet disagree on its value - does $s(7) = 2$ (from the first case), or does $s(7) = 7 + s(6) = 7 + 2 = 9$ (from the second case)?

(Note that all of the functions are technically well-defined if you restrict the domain far enough. For example, g is a well-defined function on the domain $\{6\}$, but that is not at all a "sensible" domain for a function whose definition claims to have an $n > 7$ case.)

Problem 12.2. in Discussion Manual

(b)
- $f(n) = 5f(n-1) + 1$
- $f(n) = 5(5f(n-2) + 1) + 1 = 5^2 f(n-2) + 5 + 1$
- $f(n) = 5(5(5f(n-3) + 1) + 1) + 1 = 5^3 f(n-3) + 5^2 + 5 + 1$

Based on the above, we predict the general form is that for any k,

$$f(n) = 5^k f(n-k) + \sum_{i=0}^{k-1} 5^i$$

The base case occurs when $n - k = 0$, i.e. when $k = n$:

$$f(n) = 5^n f(0) + \sum_{i=0}^{n-1} 5^i = \sum_{i=0}^{n-1} 5^i = \frac{5^n - 1}{4}$$

(c)
- $T(n) = 3T(\frac{n}{3}) + 13n$
- $T(n) = 3(3T(\frac{n}{3^2}) + 13\frac{n}{3}) + 13n = 3^2 T(\frac{n}{3^2}) + 13n + 13n$
- $T(n) = 3^2(3T(\frac{n}{3^3}) + 13\frac{n}{3^2}) + 13n + 13n = 3^3 T(\frac{n}{3^3}) + 13n + 13n + 13n$

Based on the above, we predict the general form is that for any k,

$$T(n) = 3^k T(\frac{n}{3^k}) + k \cdot 13n$$

The base case occurs when $\frac{n}{3^k} = 1$, i.e. when $k = \log_3(n)$:

$$T(n) = 3^{\log_3(n)} T(1) + \log_3(n) \cdot 13n = 47n + 13n \log_3(n)$$

Friday

Problem 12.1. in Discussion Manual

(b) The first few values are:

$g(0) = 0 = \frac{3-0-3}{4}$

$g(1) = 1 + 3(0) = 1 = \frac{9-2-3}{4}$

$g(2) = 2 + 3(1) = 5 = \frac{27-4-3}{4}$

$g(3) = 3 + 3(5) = 18 = \frac{81-6-3}{4}$

Proof that $g(n) = \frac{3^{n+1}-2n-3}{4}$ by induction on n:

Base: $g(0) = \frac{3-0-3}{4} = 0$ ✓

Induction: Suppose (as our Inductive Hypothesis) that for any $n = 1...k-1$, $g(n) = \frac{3^{n+1}-2n-3}{4}$. Then, our goal is to show that $g(k) = \frac{3^{k+1}-2k-3}{4}$. We know by the definition of g that $g(k) = k + 3g(k-1)$. Since $n = k-1$ is covered by the IH, we know $g(k) = k + 3(\frac{3^k-2(k-1)-3}{4}) = k + 3(\frac{3^k-2k-1}{4}) = k + \frac{3*3^k-6k-3}{4}$. We can combine this into one fraction as $g(k) = \frac{4k+3*3^k-6k-3}{4} = \frac{3^{k+1}-2k-3}{4}$, which is what we wanted to show.

(d) The first few values are:

$x_1 = 1$

$x_2 = 7$

$x_3 = 7x_2 - 12x_1 = 7 \cdot 7 - 12 \cdot 1 = 37$

$x_4 = 7x_3 - 12x_2 = 7 \cdot 37 - 12 \cdot 7 = 175$

Proof that $x_n = 4^n - 3^n$ by induction on n:

Base: $x_1 = 1 = 4^1 - 3^1$ and $x_2 = 7 = 4^2 - 3^2$ ✓

Induction: Suppose (as our Inductive Hypothesis) that for any positive $i < k$, $x_i = 4^i - 3^i$. We know by the definition of the sequence that $x_k = 7x_{k-1} - 12x_{k-2}$. Since $k > 2$, $k-1$ and $k-2$ are both positive so we can apply the inductive hypothesis to get $7x_{k-1} - 12x_{k-2} = 7(4^{k-1} - 3^{k-1}) - 12(4^{k-2} - 3^{k-2})$. Finally, we simplify the right hand side as follows: $7(4^{k-1} - 3^{k-1}) - 12(4^{k-2} - 3^{k-2}) = (7 \cdot 4^{k-1} - 7 \cdot 3^{k-1}) - (3 \cdot 4^{k-1} - 4 \cdot 3^{k-1}) = 4^k - 3^k$. So $x_k = 4^k - 3^k$.