Discussion Solutions Week 3

CS 173: Discrete Structures

Tuesday

Problem 5.1. in Discussion Manual

(b) This is true: 0 is a value for x such that, for every possible value of y, $xy = x$. *(The domain is very important here - if instead it were e.g. positive integers, then the statement would be false.)*

(c) This is true: every real number x has a decimal expansion. If we truncate this expansion to two decimal places, we will have a rational number y. Then, x and y will differ by at most 0.009999..., which is less than or equal to 0.01.

Problem 5.3. in Discussion Manual

(b) Let f be a function from $\mathbb{Z}^2$ to $\mathbb{Z}$, and let z be an arbitrary element in $\mathbb{Z}$ (the co-domain of f). We need to find a pre-image for z in $\mathbb{Z}^2$. Let's call that pre-image (a, b), and suppose $a = 1$ and $b = z - 27$. We know 1 is an integer, and $z - 27$ is also an integer since z and 27 are both integers. So, $(a, b) \in \mathbb{Z}^2$. Then, $f(a, b) = f(1, 27 - z) = 1 * (z - 27) + 27 = z$. Thus, we have found a pre-image to z, and f must be onto.

Problem 7.1. in Discussion Manual

(b) Let h be the given function and let x and y be elements of $\mathbb{N}$. Let's suppose $h(x) = h(y)$. Then, by the definition of h, $x^2 + 27 = y^2 + 27$. Simplifying, we get $x^2 = y^2$. Since x and y are both natural numbers, it then must be the case that $x = y$. Thus, we have proven that h is one-to-one.

Problem 7.3. in Discussion Manual

(a) Suppose that $f \circ g$ is onto and f is one-to-one. Let y be an element of B, and consider $z = f(y)$. Since $f \circ g$ is onto, there is an element $k \in A$ such that $(f \circ g)(k) = z$, i.e. $f(g(k)) = z$. Since f is one-to-one and $f(g(k)) = z = f(y)$, it follows that $g(k) = y$. Thus we have found an element of A which maps to y, so g is onto.

(b) Let $A = \{a\}$, $B = \{b, z\}$, $C = \{c\}$. Let f and g be the constant functions $f(x) = c$ and $g(x) = b$. Then $f \circ g$ is onto (because $f(g(a)) = c$), but g is not onto (because there is no input which gives the output z).

Wednesday

Problem 8.4. in Discussion Manual

- K_n: 1, unless $n = 1$, then 0. K_1 has just one vertex (which is at distance 0 from itself), so it has diameter 0. For any larger complete graph, any two distinct nodes are at distance 1 because there is an edge from every node to every other. The diameter is thus 1 (regardless of how large n is).
- C_n: $\lfloor \frac{n}{2} \rfloor$. For even n, the maximum distance is $\frac{n}{2}$, i.e. the distance between two nodes that are exactly opposite each other. For odd n, the maximum distance is still between nodes that are as close to opposite as possible, but those nodes aren't quite opposite and the two paths between them are of lengths $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. The distance between them, and hence the diameter of the graph, is the smaller of the two, $\lfloor \frac{n}{2} \rfloor$. Thus in both even and odd cases, the diameter can be expressed as $\lfloor \frac{n}{2} \rfloor$. *There are other ways to express this answer, including:*

$$\begin{cases} n/2 & \text{when } n \text{ is even} \\ (n-1)/2 & \text{when } n \text{ is odd} \end{cases}$$

- W_n: 1 if $n = 3$, 2 otherwise. For $n = 3$, W_n is just K_4 which as we saw above has diameter 1. For any larger n, there are non-adjacent nodes in the rim so the diameter must be larger than 1, but there is a short path from any node to any other that goes through the 'hub' of the wheel, so the diameter is 2.

Problem 8.5. in Discussion Manual

1. One possible circuit is $ablefijmcdkgh$.
2. No Euler circuit is possible because there is at least one node (S) with odd degree. *(Note that there* does *exist an "Euler walk" - it is possible to start from S and end at the one other odd-degree vertex. But an Euler* circuit *must start and end on the same node.)*

Problem 9.1. in Discussion Manual

(b) No isomorphism is possible: in B_1 there are two vertices of degree 3 (B and D) and they are not adjacent, while in B_2 there are also two degree-3 vertices (3 and 6) but they *are* adjacent.

(There are other features you could use to prove non-isomorphism. For example, in B_2 the three nodes of degree 4 $(1, 4, 5)$ are all adjacent to each other; B_1 also has three nodes of degree 4 (F, E, A) but there is no edge between A and F.)

Thursday

Problem 10.1. in Discussion Manual

(b) We will proceed by proving that each of the two sets is a subset of the other.

Subclaim: $X \subseteq Y$. Proof: Let z be an element of X. Then by definition of X, $z = 10x + 15y$ for some integers x, y. Factoring out the 5, we get $z = 5(2x + 3y)$. $2x + 3y$ is an integer since x, y are integers, so $z \in Y$.

Subclaim: $Y \subseteq X$. Proof: Let w be an element of Y. Then $w = 5k$ for some integer k. Then notice that $w = 10(-k) + 15k$. Since k is an integer, $-k$ is also an integer, so we see $w \in X$.

Since each set is a subset of the other, the two sets are equal.

Problem 10.2. in Discussion Manual

(c) The chromatic number is 2.

Upper bound argument: the nodes can be colored as follows: 1 is red, 4 is blue, 6 is red, 5 is blue, 3 is red, and 2 is red. Since the graph can be colored with 2 colors, the chromatic number is at most 2.

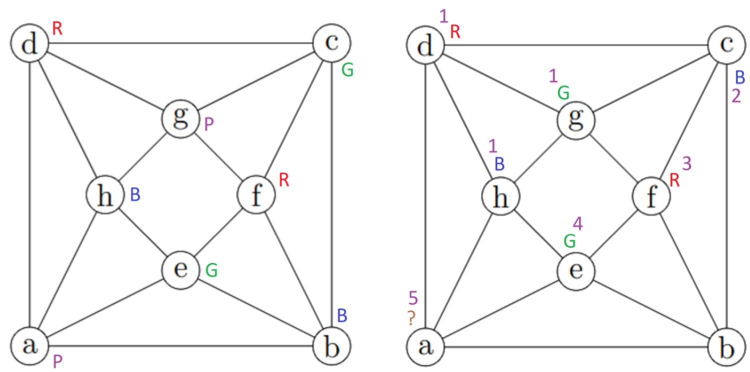
Lower bound argument: Any two nodes connected by an edge require two colors; take 1-4 for example. Since those two nodes need 2 colors, the chromatic number is at least 2.

Commentary: In this case, the upper and lower bound arguments can be combined by *carefully* coloring the graph, showing that each color choice is *forced*. Take the following as an example.

We can start by coloring the C_4 5-1-4-6. This subgraph needs at least 2 colors; we can choose red for 6 and 1, and blue for 5 and 4. Since node 3 is connected to both 5 and 4, it needs to be a different color; we can reuse red. The same is true for 2; we can also color it red. Thus the entire graph can be colored with at most 2 colors, and we showed that we need at least 2 via a careful coloring. Thus, the chromatic number is 2.

(d) Claim: the chromatic number $\chi(D)$ is 4. Proof: Figure 1a provides an upper bound of 4 by showing an explicit four-coloring, so it remains to show that the graph cannot be colored with 3 colors. We prove this as follows (see Figure 1b for a visualization): (1) Any 3-coloring must assign different colors to d, g, h; without loss of generality we call those three colors Red, Green, and Blue, respectively. (2) c is adjacent to d (Red) and g (Green), so it must be Blue. (3) f is adjacent to c (Blue) and g (Green), so it must be Red. (4) e is adjacent to f (Red) and h (Blue), so it must be Green. (5) Finally, a is adjacent to nodes of all three colors (d, h, e), so there is no possible color for a.

Notice that it would not *be enough to just have argued that* one particular *attempt at coloring with three colors didn't work. Instead, we argued that* every *attempt at three-coloring would run into this problem. At the beginning we do assign colors of our choice to d,g,h, but because color names are interchangeable and those three do need to be different from each other in any coloring, we haven't really made a significant choice, hence the "without loss of generality". For contrast, we could* not *have started the proof by assigning d,c,b to three different colors, because then we would not have addressed any colorings that may exist where d and b are given the* same *color.*



(a) A four-coloring of the graph.

(b) There is no three-coloring. (numbers correspond to steps in the proof)

Figure 1: Problem 10.2d