Discussion Solutions Week 2

CS 173: Discrete Structures

Tuesday

Problem 2.1. in Discussion Manual

(a) A few answers that keep it small are $-31, -16, -1, 14, 29, 44$. *(You can also create arbitrarily large answers, like* 1500000014*).*

(b) $[7] + [14] * [3] = [7] + [42] = [7] + 12] = [19] = [4]$. Alternatively, you can compute this more easily by getting comfortable with negative representatives: $[7] + [14] * [3] = [7] + [-1] * [3] = [7] + [-3] = [4]$

(c)

$$
\begin{aligned}
[5]^1 &= [5] \\
[5]^2 &= [5]^1 * [5] = [5] * [5] = [25] = [4] \\
[5]^3 &= [5]^2 * [5] = [4] * [5] = [20] = [6] \\
[5]^4 &= [5]^3 * [5] = [6] * [5] = [30] = [2] \\
[5]^5 &= [5]^4 * [5] = [2] * [5] = [10] = [3] \\
[5]^6 &= [5]^5 * [5] = [3] * [5] = [15] = [1]
\end{aligned}
$$

(d)

$$
\begin{aligned}
[9]^2 &= [9] * [9] = [81] = [4] \\
[9]^4 &= [9]^2 * [9]^2 = [4] * [4] = [16] = [5] \\
[9]^8 &= [9]^4 * [9]^4 = [5] * [5] = [25] = [3]
\end{aligned}
$$

So $[9]^{12} = [9]^8 * [9]^4 = [5] * [3] = [15] = [4]$.

Wednesday

Problem 3.2. in Discussion Manual

Let z be an (arbitrary) element of A. Then by the definition of A, $z = (i, j)$ for some real numbers i and j where $i^2 + j^2 \leq 1$. Since squares are non-negative, this gives us $i^2 \leq 1$ and $j^2 \leq 1$, which in turn gives us $|i| \leq 1$ and $|j| \leq 1$. Finally, this means that $z = (i, j) \in B$. Since z was an arbitrary element of A, we have shown that *every* element of A is an element of B, so $A \subseteq B$.

Problem 3.3. in Discussion Manual

(b) Proof of the claim: Let z be an element of $(A - C) - (B - C)$. Then by the definition of set subtraction, $z \in (A - C)$ and $z \notin (B - C)$. From $z \in (A - C)$, we get $z \in A$ and $z \notin C$. From $z \notin (B - C)$, we get ($z \notin B$ OR $z \in C$). Since we have established $z \notin C$, the OR statement gives us $z \notin B$. That, combined with our $z \in A$, gives us $z \in (A - B)$. Since z was arbitrary, this means that *every* element of $(A - C) - (B - C)$ is also an element of $(A - B)$, so $(A - C) - (B - C) \subseteq (A - B)$.

Problem 2. in Sets warmup

(a) Only D has more than two elements: its three elements are

- $\{2\}$
- $\{4, 5\}$
- $\emptyset$

Commentary: Note that $\{4, 5\}$ is itself only one element, and that $C = B$, so B and C each only have 2 elements.

(b) These are true: $2 \in A$, $\{2\} \in B$, $\emptyset \in D$, $\emptyset \subseteq A$, $\{2\} \subseteq A$; the remaining statements are false. *(Notice that for any object x and set Y, $x \in Y$ will always have the same truth value as $\{x\} \subseteq Y$)*

Thursday

Problem 17.2. in Discussion Manual

(a) This is a subset of the powerset of C where each element is a set of even length:

$\{\emptyset, \{Elm, Vine\}, \{Elm, Birch\}, \{Elm, Maple\}, \{Vine, Birch\}, \{Vine, Maple\},$
$\{Birch, Maple\}, \{Elm, Vine, Birch, Maple\}\}$

(b) $\{\{Elm\}, \{Pine\}\} \cap \{\emptyset, \{Elm\}, \{Maple\}, \{Elm, Maple\}\} = \{\{Elm\}\}$

(c) There are 12 elements in $C \times D$, so there are 2^{12} elements in its powerset.

(d) There are 0 elements in $B \cap D$, so there are $2^0 = 1$ elements in its powerset, *i.e.,* $\{\emptyset\}$.

Friday

Problem 17.5. in Discussion Manual

(a) Use "combinations with repetition" formula with $k = 11$ objects and $n = 3$ types: $\binom{11+3-1}{11} = \binom{13}{11}$. (Or $\binom{13}{2}$, or $\frac{13\cdot 12}{2\cdot 1} = 78$.) *WARNING: Make sure you understand how to re-derive the formula for combinations with repetition, using the stars and dividers picture (section 18.6 in the textbook). Blindly memorizing the final formula leaves you open to a range of off-by-one errors.*

(d) As given in the hint, let us consider the strings with 0, 1, 2, and 3 T's.

0 T's: 0 options. Our string must start with ST.
1 T's: We have 12 characters total and the first 2 must be ST. Then the other 10 must be chosen from 25 letters in the alphabet (excluding T). This gives us 25^{10} possible strings.
2 T's: We have 12 characters total and the first 2 must be ST. We know one of the other letters must be a T, and there are 10 places to put it. Then the other 9 must be chosen from 25 letters in the alphabet (excluding T). This gives us $10 * 25^9$ possible strings.
3 T's: We have 12 characters total and the first 2 must be ST. We know two of the other letters must be T's, and there are 10 places to put them. This gives us $\binom{10}{2} = 45$ ways to place the two T's. Then the other 8 characters must be chosen from 25 letters in the alphabet (excluding T). This gives us $45 * 25^8$ possible strings.

Then, in total, we have $25^{10} + 10 * 25^9 + 45 * 25^8$ possible strings that start with ST and contain 3 or fewer Ts.

(e) We can sum the possible combinations of each odd-sized set: $\binom{10}{1} + \binom{10}{3} + \binom{10}{5} + \binom{10}{7} + \binom{10}{9}$