Discussion Solutions Week 1

CS 173: Discrete Structures

Monday

Problem 1.5. in Discussion Manual

1. Yes, the set of operators $\{\vee, \neg\}$ is functionally complete, as we can express $\wedge$ in terms of these operators:

$$\alpha \wedge \beta \equiv \neg(\neg\alpha \vee \neg\beta)$$

So in any formula involving $\wedge$, we can replace subformulas involving $\wedge$ systematically using the above equivalence to get an equivalent formula involving only $\neg$ and $\vee$.

2.

$$\neg p \equiv p \uparrow p$$

When $p = T$, $p \uparrow p = F = \neg p$.
When $p = F$, $p \uparrow p = T = \neg p$.
Hence the above equivalence holds. The following truth table also verifies equivalence:

p	$\neg p$	$p \uparrow p$
T	F	F
F	T	T

Alternate solution: $p \uparrow T$ also is equivalent to $\neg p$,

3. Notice that the truth table entries for $p \uparrow q$ are precisely the negation of what we need, i.e, for $p \vee q$, provided p and q are negated before combining with $\uparrow$.

So we conjecture:

$$p \vee q \equiv (\neg p) \uparrow (\neg q)$$

Since we know how to express negation using $\uparrow$ (as argued above), we conjecture:

$$p \vee q \equiv (p \uparrow p) \uparrow (q \uparrow q)$$

The truth table below verifies this:

p	q	$p \uparrow p$	$q \uparrow q$	$(p \uparrow p) \uparrow (q \uparrow q)$	$(p \vee q)$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

Alternate Solution: There are many solutions to this problem, of course. Here's an alternative that uses the alternative encoding of $\neg$, as mentioned above, which uses the *constant* T (true).

$$p \vee q \equiv (p \uparrow T) \uparrow (q \uparrow T)$$

This too can be verified using a truth table.

4. The set of operators $\{\uparrow\}$ is functionally complete since we can take any formula involving operators in $\{\neg, \vee\}$ and replace subformulas systematically using equivalent formulas that use only the $\uparrow$ operator. Since the set of operators $\{\neg, \vee\}$ is functionally complete, so is $\{\uparrow\}$.

5. Let us try to express $\uparrow$ using $\downarrow$.

   First, notice that $p \uparrow q \equiv \neg(p \wedge q)$.
   And $p \downarrow q \equiv \neg(p \vee q)$.

   So $p \uparrow q$
   $\equiv \neg(p \wedge q)$
   $\equiv \neg p \vee \neg q$
   $\equiv \neg(\neg(\neg p \vee \neg q))$
   $\equiv \neg((\neg p) \downarrow (\neg q))$ So what's left is to express negation using $\downarrow$. Notice that $\neg r \equiv (r \downarrow r)$. Using this to rewrite negation gives: $p \uparrow q$
   $\equiv \neg((p \downarrow p) \downarrow (q \downarrow q))$
   $\equiv ((p \downarrow p) \downarrow (q \downarrow q)) \downarrow ((p \downarrow p) \downarrow (q \downarrow q))$ (Verify the above using truth tables!) Since we have expressed $\uparrow$ using $\downarrow$ operator only, and given that $\{\uparrow\}$ is functionally complete, it follows that $\{\downarrow\}$ is functionally complete as well.

   Alternate Solution: Another way to express $\uparrow$ using $\downarrow$ is:

$$((p \downarrow F) \downarrow (q \downarrow F)) \downarrow F$$

Tuesday

Problem 1. Logical Reasoning

First, we will model the given statements using (first-order) logic. *(This translation isn't strictly necessary, but in many cases it may be easier to see what's going on, and the concise statements may be easier to work with.)* We'll use the following predicates, where variables range over animals:

- $c(x)$: x is chestnut-eating
- $fl(x)$: x is fun-loving
- $penguin(x)$: x is a penguin
- $m(x)$: x eats mulberries
- $wd(x)$: x is well-dressed
- $comfy(x)$: x is comfortable

We can then formulate the facts as follows. Let us write them using *implications* as much as possible.

1. $\forall x.\ c(x) \Rightarrow fl(x)$
2. $\forall x.\ penguin(x) \Rightarrow (\neg m(x))$
3. $\exists x.\ wd(x) \wedge \neg comfy(x)$
4. $\exists x.\ penguin(x) \wedge \neg comfy(x)$
5. $\forall x.\ m(x) \vee c(x)$, which is equivalent to $\forall x.\ (\neg m(x)) \Rightarrow c(x)$ *(Note: this is* not *the same as* $\forall x.m(x) \vee \forall x.c(x)$*)*
6. $\neg \exists x.\ (\neg comfy(x) \wedge m(x))$, which is equivalent to $\forall x.\ (comfy(x) \vee \neg m(x))$, which is equivalent to $\forall x.\ (m(x) \Rightarrow comfy(x))$.

(a) Every comfortable penguin eats mulberries.

(It is tempting to say simply that the statement must be false because it directly contradicts Fact 2. However, notice that if the world contains zero comfortable penguins, this statement would actually be true.)

Let us construct a concrete counterexample to the statement. We'll start with a comfortable penguin called p that does *not* eat mulberries. Let's see if we can build a world with p that is consistent with all known facts.

(The counterexample you come up with does not have to be the same as ours - some of the choices we make below are forced by the facts, but others are up to you.)

(2) says no penguin eats mulberries. This is fine for p as it does not eat mulberries. (5) is also consistent, but in order to satisfy (5), we must allow that p eats chestnuts. (6) is consistent with our assumptions, since p is anyway comfortable, and (6) talks about uncomfortable animals only. (1) talks about chestnut-eating animals, and so we must allow that p is fun-loving. To be consistent with (4), we'll have to add a second penguin, p', which is uncomfortable. p' does not

eat mulberries, eats chestnuts, is fun-loving (to satisfy (4)), and is not well-dressed. Finally, to be consistent with (3), we'll declare that p is not well-dressed, but then we need to add one more animal a. a is not a penguin, a is uncomfortable (to satisfy (3)). We can have this animal not eat mulberry (to satisfy (6)), and eat chestnuts (to satisfy (5)), and be fun-loving (to satisfy (1)).

So overall, our world W has exactly three animals, p, p', and a, which can be described as follows:

- $\mathit{penguin}(p)$, $\mathit{comfy}(p)$, $\neg m(p)$, $c(p)$, $fl(p)$, $\neg wd(p)$.
- $\neg\mathit{penguin}(a)$, $\neg\mathit{comfy}(a)$, $\neg m(a)$, $c(a)$, $fl(a)$, $wd(a)$.
- $\mathit{penguin}(p')$, $\neg\mathit{comfy}(p')$, $\neg m(p')$, $c(p')$, $fl(p')$, $\neg wd(p')$.

We see that this world satisfies all 6 facts but does not satisfy the given statement (which is, formally, $\forall x.(\mathit{penguin}(x) \land \mathit{comfy}(x)) \Rightarrow m(x))$.

Hence the given statement does not logically follow from the facts.

(b) At least one penguin is well-dressed.

The world W we constructed for the part (a) shows that there need not be a well-dressed penguin. So this statement is not entailed by the facts.

(The only fact that requires a penguin to exist at all is (4). That penguin is uncomfortable, but it needn't be well-dressed.)

(c) No penguins are fun-loving.

Again, the world W we painted above shows that there can be fun-loving penguins. So the statement is not entailed by the facts.

(d) There is at least one fun-loving well-dressed animal.

This is provable.

- From (3), we know there is at least one well-dressed animal that is also uncomfortable; call this animal a.
- From (6), we know that a does not eat mulberries.
- From (5), we know that a must eat chestnuts.
- From (1), we know that a must be fun-loving.
- Hence there is at least one fun-loving well-dressed animal.

(e) All penguins are fun-loving.

- Let p be any penguin.
- From (2), we know that p does not eat mulberries.
- From (5), we know that p must eat chestnuts.
- From (1), we know that p must be fun-loving.

Note that the above argument is true even if there are no penguins. Of course, (4) says there is at least one penguin. But we didn't use (4) for the above argument.

(f) All penguins are uncomfortable.

This statement doesn't seem to hold as it seems all penguins are in fact comfortable. In fact, the statement is false in the world W we created for (a) (where p is a comfortable penguin). So the statement is not entailed by the facts.

Problem 1.2. in Discussion Manual

(b) This universal claim is false, as demonstrated by the following counterexample. Suppose that $p = 1$ and $q = 2$. Then $(p+q)^2 = 9$, but $p^2 + q^2 = 5$, so $(p+q)^2 \neq p^2 + q^2$.

Commentary: There are many possible answers; you do not need to pick the "smallest" counterexample but you should pick one that is easy to follow - e.g. $p = 13, q = 54$ is technically correct but a bad choice. It is also technically possible to classify all *possible counterexamples (e.g. something like "The claim is false, as shown by any p and q that are both non-zero."), but you should* **not** *attempt to do so in your solution since providing a single counterexample is sufficient and much easier.*

(c) The claim is not true. Suppose that w, x, y, z are $0, 1, -1000, -5$, respectively. These satisfy the conditions that $w < x$ and $y < z$. However, $wy = 0 > -5 = xz$, so $wy \not< xz$.

Commentary: Again there are many answers, but they may be harder to find than in part (b). I found this answer by checking what happens if $w = 0$. Checking "edge cases" like that is often a good place to start because you may find that things simplify/cancel - in this case, it becomes a search for x, y, z where $0 < x$, $y < z$, $0y = 0 \geq xz$, and suddenly I have lots of freedom to choose y since it disappeared from the final inequality, and I also have clear direction of how to continue since I see x must be positive while xz should be negative (or zero).

Problem 1.3. in Discussion Manual

(a) For this problem we will do a proof by cases.

Let x be an integer, and let $|x+7| > 8$. We have two cases, since $|x+7| > 8$ can be rewritten as an "or" statement (*i.e.*, $x + 7 > 8$ ***or*** $x + 7 < -8$).

Case 1: $x + 7 > 8$. Then, it follows that $x > 1$. If $x > 1$, the absolute value of x must also be greater than 1, so we have $|x| > 1$.

Case 2: $x + 7 < -8$. Then, it follows that $x < -15$. If $x < -15$, it must also be the case that $x < -1$. This implies, like in the previous case, that $|x| > 1$.

In both cases we show that $|x| > 1$, so we have proven the claim.

Wednesday

Problem 1.1. in Discussion Manual

(a) The negation should be constructed as follows:

It is not the case that if my plant is dead, then I didn't water it or I left it in the dark.

My plant is dead and it's not the case that I didn't water it or I left it in the dark.

My plant is dead and I watered it and I didn't leave it in the dark.

The contrapositive is as follows:

If I watered my plant and it didn't leave it in the dark, then it is not dead.

(c) The negation should be constructed as follows:

It is not the case that for every martian w, if w is green then w is tall or w is ticklish.

There exists a martian w such that it is not the case that if w is green then w is tall or w is ticklish.

There exists a martian w such that w is green and it is not the case that w is tall or w is ticklish.

There exists a martian w such that w is green and w is not tall and w is not ticklish.

The contrapositive is as follows:

For every martian w, if w is not tall and w is not ticklish, then w is not green.

Problem 1.3. in Discussion Manual

(c) We will proceed by proving the contrapositive, *i.e.,* we will show that for all integers m and n, if m is odd and n is odd, then mn is odd. So let m and n be odd integers. By the definition of odd integers, m can be written as $2k+1$ where k is an integer, and n can be written as $2j+1$ where j is an integer. Then, $mn = 4kj+2k+2j+1$. This can be written as $2(2kj+k+j)+1$. Since k and j are integers, we know that $2kj+k+j$ is also an integer, let's call it h. Then, we have $mn = 2h+1$. Since h is an integer, mn is odd, so we have proven the contrapositive.

(d) We will proceed by proving the contrapositive, *i.e.,* we will show that for every real number x, if $x < 2$ and $x \geq 1$ then $x^2 - 3x + 2 \leq 0$. So let x be a real number and suppose that $x < 2$ and $x \geq 1$. Since $x < 2$, $x - 2 < 0$. Since $x \geq 1$, $x - 1 \geq 0$. Therefore $(x-1)(x-2) \leq 0$. So $x^2 - 3x + 2 = (x-1)(x-2) \leq 0$, which is what we needed to prove.

Friday

Problem 2.2. in Discussion Manual

Common misconception: *You cannot solve parts a and b by setting up a system of two equations with just two variables, e.g. in part b it is not accurate to say "$x = 6a + 5$ and also $x = 10a + 3$". It is technically accurate (but confusing) to say "$\exists a, x = 6a + 5$, and also $\exists a, x = 10a + 3$" because the quantifiers make clear the separate scopes, but best to just use separate variable names (i.e. it is correct to say "there are a and b for which $x = 6a + 5$ and $x = 10b + 3$").*

(a) There is no such x. From the first congruence, we would need $x = 7 + 9p = 1 + 3(2 + 3p)$ for some integer p, so $remainder(x, 3) = 1$. From the second congruence, we would need $x = 5 + 12q = 2 + 3(1 + 4q)$ for some integer q, so $remainder(x, 3) = 2$. There is no number which has two different remainders when divided by 3.

Commentary: How did we know to take remainders at all? They're frequently useful once you're already in the world of modular arithmetic, so worth a try. How did we know to take remainders dividing by 3 specifically? Notice that if all you have is the congruence $x \equiv 5 \pmod{12}$, then you actually can't determine the value for many remainders, like $remainder(x, 7)$. (Try it!) Given congruences mod 9 and 12, $remainder(x, 3)$ is the only remainder that can be calculated from both of them.

Alternate solution: There is no such x. From the first congruence, we would need $x = 7+9p$ for some integer p, and from the second congruence, we would need $x = 5 + 12q$ for some integer q, so $7 + 9p = 5 + 12q$. This rearranges to $3p - 4q = \frac{2}{3}$. But then there are no possible values for p and q, since the left hand side will always be an integer while the right is not.

(b) Yes, for example $x = 23$. $23 = 6 \cdot 3 + 5$, and also $23 = 10 \cdot 2 + 3$.

Problem 2.3. in Discussion Manual

(a) This is false. Let's take $p = 10$, $q = 7$, $r = 5$. Then $gcd(10, 7) = 1$, and $gcd(7, 5) = 1$, but $gcd(10, 5) = 5 \neq 1$.

Problem 2.4. in Discussion Manual

(a) Let a, b, c be integers. Also assume $a|b$ and $b|c$. By the definition of divides, we have $b = am$ for some integer m, and $c = bn$ for some integer n. Plugging the first equation into the second, we have $c = amn$. Since m and n are integers, mn is also an integer. Let's say $mn = p$. Then, $c = ap$ where p is an integer, so $a|c$.