Thursday 7/31: Countability pt. 1

In previous classes we have discussed various infinite sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. We know that these sets are infinite; but that doesn't mean they have the same *cardinalty*!

Recall that two sets A and B have the same **cardinality** ($|A| = |B|$) iff there is a bijection from A to B.

bijection example: Provide a bijection that shows $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

Consider a function $f : \mathbb{N} \to \mathbb{Z}$ such that $f(n) = n/2$ when n is even, and $f(n) = -(n+1)/2$ when n is odd. Note that when we input natural numbers, the even piece of the piecewise function produces outputs $\{0, 1, 2, 3..\}$ and the odd piece produces $\{-1, -2, -3..\}$. So we can see that every element in the co-domain is covered, and there are no repeats. So our function is both onto and one-to-one—making it a bijection!

Because we found a bijection, $|\mathbb{N}| = |\mathbb{Z}|$.

An infinite set A is **countably infinite** iff there is a bijection from $\mathbb{N}$ to A. Intuitively, if we can find this bijection, $\mathbb{N}$ serves as an index for the set A, so we can put A in some *order* in which we can *count* it.

But what if it's too difficult to produce a bijection?

Cantor Schroeder Bernstein Theorem: $|A| \leq |B|$ iff there exists a one-to-one function from A to B (think about what you know about one-to-one functions, this should follow directly from the definition!).

So, if we find a one-to-one function $f : A \to B$ and another $g : B \to A$, then $|A| \leq |B|$ and $|B| \leq |A|$. So by a two-way bounding argument, $|A| = |B|$.

Another bijection example: Provide two one-to-one functions that show the non-negative rationals are countably infinite.

First, we have $f : \mathbb{N} \to \mathbb{Q}^{\geq 0}$ such that $f(n) = n$.

Next, we have $g : \mathbb{Q}^{\geq 0} \to \mathbb{N}$ such that $g(\frac{x}{y}) = 2^x 3^y$. Here, $\frac{x}{y}$ is the rational number in lowest terms. Since all natural numbers have a unique prime number decomposition, we know we will never get the same natural number out as output twice.

Thus, $\mathbb{Q}^{\geq 0}$ is countably infinite.

Tomorrow we'll talk about uncountable sets...