Friday 8/1: Countability pt. 2

Yesterday we looked at how to prove sets are countable—specifically, we use the term countable to mean either finite or countably infinite. But what if they aren't? If we can index an infinite set by $\mathbb{N}$, it is countable. So, if it is impossible to index that set by $\mathbb{N}$, it is uncountable.

Today we're going to prove that $\mathbb{P}(\mathbb{N})$ is uncountable. Before we get into the explanation, let's talk about how we're going to represent each element of the powerset. Recall that $\mathbb{N} = \{0, 1, 2, 3...\}$. Each element of $\mathbb{P}(\mathbb{N})$ can be represented by an infinitely long bit string, where a 0 means the element of that index is not in the subset, and 1 means it is.

- 010001.... means 1 and 5 are in the set, so this is the element $\{1, 5...\}$
- 111000.... means 0,1 and 2 are in the set, so this is the element $\{0, 1, 2...\}$

Since the natural numbers are an infinite set, each bit string is infinitely long.

Now, let's proceed with a proof by contradiction. Say, instead, that $\mathbb{P}(\mathbb{N})$ is countable. If this is true, I should be able to write the elements in an ordered list indexed by $\mathbb{N}$. Here is a list of what I claim is *all* the bit strings:

	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	...
v_0	1	1	0	1	1	0	1	1	1	1	...
v_1	1	1	0	0	1	0	1	1	0	0	...
v_2	0	0	0	0	1	0	0	1	0	0	...
v_3	0	1	1	1	1	0	1	0	0	0	...
v_4	0	0	0	0	1	1	1	0	1	1	...
v_5	1	1	1	0	1	0	1	0	0	1	...

However, this can't be the entire list! Because I can produce a new bit string not in the list, because it differs by each string v_k at index k. In this case, I know 001001... is not in the list.

Since the infinite bit strings cannot be written in a list indexed by $\mathbb{N}$, this set is uncountable. It also represents the set $\mathbb{P}(\mathbb{N})$, which is uncountable as well.

Note that we can use a similar argument to show that the set of real numbers in $[0, 1)$ is uncountable! And if that set is uncountable, so is the larger set $\mathbb{R}$. Try this proof at home.

A few things to note about uncountable sets:

- if $A \subseteq B$ and A is uncountable, then B is also uncountable
- if A and B are both uncountable, it is not necessarily the case that $|A| = |B|$. There are different sizes of uncountable sets, but all countable sets are the same size.
- $|A| < |\mathbb{P}(A)|$

Please review the section on uncomputability on your own