Friday 7/25: Proof by Contradiction

Proof by contradiction is based on the fact that a statement has deterministic true/false value, i.e. it is either true or false, and there is no "maybe". Then, if we can show that it is impossible for a claim to be false, by showing that it could lead to a contradiction with ground truth or assumptions in the claim itself, then we are equivalently proving that this claim is true.

Proof by contradiction is typically used to prove claims that a certain type of object cannot exist. The negation of the claim then says that an object of this sort does exist. The existence of an object with specified properties is often a good starting point for a proof. For example, suppose we are trying to prove this claim by contradiction: *there is no largest even integer.*

> Proof: Suppose not. That is, suppose that there were a largest even integer. Let's call it k.
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> Since k is even, it has the form $2n$, where n is an integer. Consider $k + 2$. $k + 2 = (2n) + 2 = 2(n + 1)$. So $k + 2$ is even. But $k + 2$ is larger than k.
>
> This contradicts our assumption that k was the largest even integer. So our original claim must have been true.

A structured proof by contradiction often starts with the line **"suppose not"**, which is a declaration that you are building a proof by contradiction (recall how we often say "proof by induction on n" for inductive proofs). You can also use other phrasing that has similar meanings. Next, you need to spell out **exactly what the negation of the claim is**, just like how you would with inductive hypothesis.

Then, you can work on the algebra or other logical reasoning until you reach a point of contradiction. You should also **point out what the contradiction is**, so that readers can fully understand your proof.

Let's take a look at another example. Suppose we know that $\sqrt{6}$ is irrational. Prove that $\sqrt{2} + \sqrt{3}$ is irrational by contradiction.

With the hint that $\sqrt{6}$ is irrational, we should work on the assumption that $\sqrt{2}+\sqrt{3}$ is rational and reach a point that makes $\sqrt{6}$ rational, so that we can have a contradiction. So how do we do that?

> Proof: Suppose not. That is, suppose that $\sqrt{2} + \sqrt{3}$ is rational.
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> Then there are integers p and q (q non-zero) such that $\sqrt{2} + \sqrt{3} = \frac{p}{q}$.
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> Squaring both sides of this equation gives $2+2\sqrt{6}+3 = \frac{p^2}{q^2}$. So $2\sqrt{6} = \frac{p^2}{q^2} - 5 = \frac{p^2-5q^2}{q^2}$. So $\sqrt{6} = \frac{p^2}{q^2} - 5 = \frac{p^2-5q^2}{2q^2}$.
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> But notice that $p^2 - 5q^2$ and $2q^2$ are both integers since p and q are integers. So this means that $\sqrt{6}$ is the ratio of two integers and therefore rational.
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> But we know that $\sqrt{6}$ is not rational. Since our original assumption led to a contradiction, $\sqrt{2} + \sqrt{3}$ must be irrational.