Thursday 7/17: Big-O

Asymptotic Relationships

We talk about the runtime of a program in terms of how long it takes to run on an input of size n, with respect to that size n.

e.g., on size n input, some program runs in n^2 time. n^2 is approximately the number of constant-time operations—*i.e.,* additions.

If I have a program with a runtime of $f(n)$ and you have a program with a runtime of $g(n)$, how do we know whose is faster? We only care about large inputs. To figure this out, we use **asymptotic relationships**. $<<$ means asymptotically smaller.

$$1 << \log n << n << n \log n << n^2 << n^2 \log n << n^3 << 2^n << 3^n << n!$$

Some algebraic rules also work here. For example, if $h(n)$ is a nonzero function, then:

$$f(n) << g(n) \rightarrow f(n)h(n) << g(n)h(n)$$

In addition, we evaluate functions' runtimes based on the dominant term:

$$3^n + 102n^3 << 27n + 2n! + 17$$

since

$$3^n << n!$$

Big-O

We can define a looser relationship than asymptotically smaller, because sometimes asymptotic relationships are not sufficient to compare functions.

big-O: a function $f(n)$ is $O(g(n))$ iff there exist $c, k \in \mathbb{R}^+$ such that $0 \leq f(n) \leq cg(n)$ for every $n \geq k$.

In other words, if $f(n) << g(n)$, then $f(n)$ is $O(g(n))$.

big-O example: Show that $3n^2 + 2n$ is $O(n^3)$.

I will choose $c = 2$. Then, I want to know for which values of n, $0 \leq 3n^2 + 2n \leq 2n^3$.

We can do the next part of the proof a bit backwards to find our k value.

$$\begin{aligned} 3n^2 + 2n &\leq 2n^3 \\ 3n + 2 &\leq 2n^2 \\ 0 &\leq 2n^2 - 3n - 2 \\ 0 &\leq (2n+1)(n-2) \end{aligned}$$

This gives us $n \geq 2$ and $n \geq -\frac{1}{2}$. So, if $n \geq 2$,

$$
\begin{aligned}
(n-2)(2n+1) &\geq 0 \\
2n^2 - 3n - 2 &\geq 0 \\
2n^2 &\geq 3n + 2 \\
2n^3 &\geq 3n^2 + 2n
\end{aligned}
$$

Which is what we wanted to show, so we can choose $k \geq 2$.

Note that I found the tightest k value for the c value I chose. This is not necessary, you can choose a looser bound on n, and still prove a big-O relationship.

$f(n)$ is $\Theta(g(n))$ iff $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$. Though $3n^2 + 2n$ is $O(n^3)$, n^3 is NOT $O(3n^2 + 2n)$.

big-Θ example: Show that $2n^2$ is $O(n^2)$.

Choose $c = 4$. Then I want to show that $2n^2 \leq 4n^2$. This is true for $n \geq 0$.
n^2 is $O(2n^2)$ and n^2 is $\Theta(2n^2)$ as well.