Wednesday 7/9: Inductions Pt.2

Use strong induction

There are two types inductive proofs: strong induction and weak induction. All examples you see in this course use strong induction, which assumes $P(n)$ is true *for all* x *= 0, 1, ...,* $k-1$ and then proves $P(k)$ for some integer k. In contrast, weak induction *simply assumes* $P(k-1)$ *is true* and then shows that $P(k)$ is true for some integer k. For the sake of robustness, you are only allowed to use strong induction in this course, because weak inductions are more error-prone. Strong inductions are especially powerful when your inductive step involves multiple "smaller problems".

How many base cases do you need?

In most cases, when you write an inductive proof, you will only need to show one base case and leave the rest for the inductive step. However, if in your inductive step (proving $P(k)$) you need more than just $P(k-1)$, then you might need to show multiple base cases.

For example, prove the following claim: *Every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.*

> Proof: by induction on the amount of postage.
> Base: If the postage is 12 cents, we can make it with three 4-cent stamps. If the postage is 13 cents, we can make it with two 4-cent stamps. plus a 5-cent stamp. If it is 14, we use one 4-cent stamp plus two 5-cent stamps. If it is 15, we use three 5-cent stamps.
> Induction: Suppose that we have show how to construct postage for every value from 12 up through $k-1$. We need to show how to construct k cents of postage. Since we've already proved base cases up through 15 cents, we'll assume that $k \geq 16$.
> Since $k \geq 16, k-4 \geq 12$. So by the inductive hypothesis, we can construct postage for $k-4$ cents using m 4-cent stamps and n 5-cent stamps, for some natural numbers m and n. In other words $k-4 = 4m+5n$. But then $k = 4(m+1)+5n$. So we can construct k cents of postage using $m+1$ 4-cent stamps and n 5-cent stamps, which is what we needed to show.

In the inductive step, proving P(k) requires P(k-4) to be true, because the smallest amount of postage to add is 4 cents. If you only show one base case, you can only go from P(12) to P(16) to P(20)..., and that inductive chain is incomplete. Therefore, for this induction to cover all integers ≥ 12, you need to **fill in the gap between the smallest base case and the smallest inductive case**.

Choosing the inductive variable

We often do induction on a variable that is an integer, has a lower bound, and goes up infinitely. In most cases, there is only one such variable in the problem. However, when there are multiple of them, it becomes tricky to decide which variable to induct on.

For example, a graph can have multiple variables satisfying the above properties, including the number of vertices, the number of edges, maximum degree, diameter, etc. The most common inductive variable for graphs is the number of vertices or edges, and we often just fix an arbitrary value for other variables.

Now, suppose we are trying to prove this claim: *For any positive integer* D*, if all nodes in a graph* G *have degree* $\leq D$*, then* G *can be colored with* $D+1$ *colors.*

You can see the full explanation in the textbook (Chapter 11.8). Note that in this question, there are two variables, the explicitly defined D, and the hidden variable *number of vertices (n)*. **The way to choose which variable to induct on depends on whether you can break down the inductive step into smaller problems where the inductive hypothesis can apply.**

For example, if you fix D and induct on n, in your IH you can define an arbitrary graph G with n vertices and max degree is D. Then, you can easily take out one vertex and the remaining graph G' has $n-1$ vertices with max degree being D. This is where you can apply the IH.

In contrast, if you fix n and want to induct on D, then you will get stuck trying to construct a G' with n vertices and max degree being $D-1$ out of G, which is an arbitrary graph with n vertices and max degree being D.

Therefore, for this question, the only way to do induction is by induction on the number of vertices in a graph.