Tuesday 7/8: Inductions Pt.1

Summations

If a_i is some formula that depends on i, then

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$$

For example

$$\sum_{i=1}^{n} i = 1 + 2 + 3... + n$$

$$\sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + ... \frac{1}{2^n}$$

$$\sum_{i=1}^{n} 1 = 1 + 1 + 1.. + 1 = n$$

Certain summations have known closed forms, which are succinct formulas that can simplify the calculation and can be taken for granted (except when you are asked to prove a closed form).

A few closed forms that you should *memorize*

$$\sum_{i=1}^{n} i = 1 + 2 + 3... + n = \frac{(n)(n+1)}{2}$$

$$\sum_{k=0}^{n} r^k = 1 + r + r^2... + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Induction outline

Suppose you are given a claim $P(n)$ and your task is to show that $P(n)$ is true for all positive integers n.

A proof using induction on n should contain the following:

0) State that you will be using proof by induction and specify the variable. For example, say something like "We will prove by induction on n"

1) Base case: Show that $P(1)$ is true. Sometimes, you might need to prove $P(2)$ and/or even $P(3)$, depending on your need for the inductive step.

2) State your inductive hypothesis (IH): *Suppose that P(n) is true for all* $n = 1...k-1$, *for some* $k \geq 2$.

3) Inductive step: Prove that $P(k)$ is true, using assumptions of IH.

Let's see the following example: Prove via induction that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for any positive integer n.

The outline/planning logic may look like the following:

> Proof: We will prove by induction on n.
> Base case: We need show $P(1)$ is true, i.e. $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$.
> Inductive hypothesis: Suppose that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ is true for $n = 1, 2, ...k-1$.
> Inductive step: Show that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ is true.

The full proof might then look like:

> Proof: We will show that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for any positive integer n, using induction on n.
>
> Base: We need to show that the formula holds for $n = 1$. $\sum_{i=1}^{1} i = 1$, and $\frac{1(1+1)}{2} = 1$. So the two are equal for $n = 1$.
>
> Inductive hypothesis: Suppose that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n = 1, 2, ..., k-1$. When stating your IH, please do not say "the claim". Spell it out with proper variables.
>
> Inductive step: We need to show that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
> By the definition of summation notation, $\sum_{i=1}^{k} i = \sum_{i=1}^{k-1} i + k$ Our inductive hypothesis states that at $n = k-1$, $\sum_{i=1}^{k-1} i = \frac{(k-1)(k)}{2}$.
> Combining these two formulas, we get that $\sum_{i=1}^{k} i = \frac{(k-1)(k)}{2} + k = \frac{k^2-k}{2} + \frac{2k}{2} = \frac{k^2+k}{2} = \frac{(k)(k+1)}{2}$, which is what we needed to show.

Why is this legit?

There are several ways to think about mathematical induction, and under- stand why it's a legitimate proof technique. Different people prefer different motivations at this point, so here are a few options.

> **Domino Theory:** Imagine an infinite line of dominoes. The base step pushes the first one over. The inductive step claims that one domino falling down will push over the next domino in the line. So dominos will start to fall from the beginning all the way down the line. This process continues forever, because the line is infinitely long. However, if you focus on any specific domino, it falls after some specific finite delay.
> **Recursion fairy:** The recursion fairy is the mathematician's version of a programming assistant. Suppose you tell her how to do the proof for P(1) and also why P(1) up through P(k) implies P($k+1$). Then suppose you pick any integer (e.g. 1034) then she can take this recipe and use it to fill in all the details of a normal direct proof that P holds for this particular integer. That is, she takes P(1), then uses the inductive step to get from P(1) to P(2), and so on up to P(1034).
> **Defining property of the integers:** The integers are set up mathematically so that induction will work. Some formal sets of axioms defining the integers include a rule saying that induction works. Other axiom sets include the "well-ordering" property: any subset that has a lower bound also has a smallest

element. This is equivalent to an axiom that explicitly states that induction works. Both of these axioms prevent the integers from having extra very large elements that can't be reached by repeatedly adding one to some starting integer. So, for example, ∞ is not an integer.