Friday 7/11: Recursive induction

Now that we have learned induction and recursion, let's see some inductive proofs with recursive definitions. For example, let's prove the following claim about the Fibonacci numbers: *For any* $n \geq 0, F_{3n}$ *is even.*

Proof: by induction on n.

Base: $F_0 = 0$, which is even. $F_3 = 2$, which is also even.

IH: Suppose that F_{3n} is even for $n = 0, 1, ..., k$.

IS: We need to show that $F_{3(k+1)}$ is even. By definition of Fibonacci numbers, we know $F_{3(k+1)} = F_{3k+3} = F_{3k+1} + F_{3k+2}$. (1)

Also note that the largest case under our inductive hypothesis is that F_{3k} is even. So what can we do about (1)? We can unroll it again.

Since $F_{3k+2} = F_{3k+1} + F_{3k}$ (by definition of Fibonacci numbers), we know $F_{3(k+1)} = F_{3k+1} + F_{3k+2} = F_{3k+1} + (F_{3k+1} + F_{3k}) = 2(F_{3k+1}) + F_{3k}$

Now we know $2(F_{3k+1})$ is even because it is a multiple of 2 (note F_{3k+1} is an integer by definition of Fibonacci numbers), and F_{3k} is even based on IH. Therefore, the sum of these two terms is also even, i.e. $F_{3(k+1)}$ is even.

Let's take a look at another example. Suppose we have a function $f : \mathbb{N} \to \mathbb{N}$ that is defined as:

$$f(0) = 2$$
$$f(1) = 3$$
$$\forall n \geq 2, f(n) = 3f(n-1) - 2f(n-2)$$

Now let's try to prove this claim: $\forall n \in \mathbb{N}, f(n) = 2^n + 1$

Proof by induction on n

Base: $f(0) = 2$, $2^0 + 1 = 1 + 1 = 2$, so $f(n) = 2^n + 1$ is true for $n = 0$. $f(1) = 3$, $2^1 + 1 = 2 + 1 = 3$, so $f(n) = 2^n + 1$ is true for $n = 1$.

IH: Suppose that $f(n) = 2^n + 1$ for $n = 0, 1, ...k$.

IS: Let's show that $f(k+1) = 2^{k+1} + 1$

Based on the recursive definition, $f(k+1) = 3f(k) - 2f(k-1)$. By IH, $f(k) = 2^k + 1$ and $f(k-1) = 2^{k-1} + 1$.

Therefore, $f(k+1) = 3f(k) - 2f(k-1) = 3(2^k + 1) - 2(2^{k-1} + 1) = 2 * 2^k + 1 = 2^{k+1} + 1$, which is what we need to show.

Note that in the IS we used $f(k)$ and $f(k-1)$, so we must use strong induction (suppose claim is true for all $n = 0, 1, ..k$) and prove two base cases ($f(0)$ and $f(1)$).