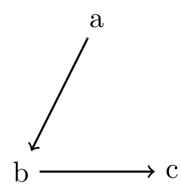
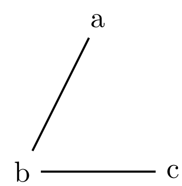
Wednesday 7/2: Graphs

Graph Overview

We define a graph $G = (V, E)$ where V is the set of vertices $v \in V$ and E is the set of edges.

For example, $V = \{a, b, c\}$. We can write the set E out in various ways. Directed: $E = \{(a, b), (b, c)\}$, or undirected $E = \{ab, bc\}$ or $E = \{\{a, b\}, \{b, c\}\}$. There are other representations as well.





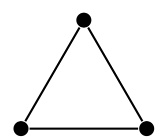
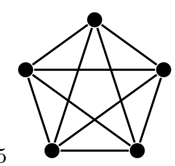
Directed Undirected

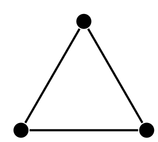
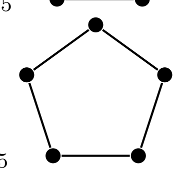
Moving forward, we are dealing with undirected graphs unless otherwise stated.

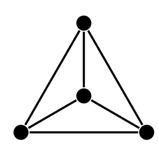
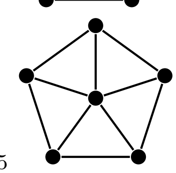
In this class we deal with *simple graphs:* which have *no* self loops and *no* multiple edges. They will also have at least one node.

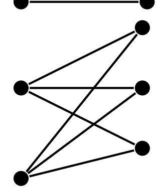
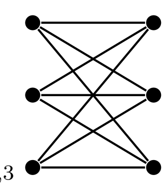
degree(v): the degree of a node is the number of edges that have v as an endpoint. Each edge contributes a total of 2 degrees to the entire graph, so,

$\sum_{v \in V} deg(v) = 2|E|$

Complete Graphs (K_n with n total nodes) K_3  K_5 

Cycle Graphs (C_n with n total nodes) C_3  C_5 

Wheel Graphs (W_n with $n + 1$ total nodes) W_3  W_5 

Complete Bipartite (K_{m+n} with $m + n$ total nodes) $K_{2,3}$  $K_{3,3}$ 

Traversing a Graph

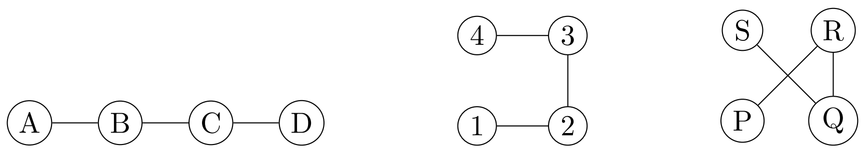
- *walk:* finite sequence of nodes from node a to b. Can be described by either a sequence of nodes or edges, and the length is the number of edges traversed. A closed walk is one that returns to the beginning (a and b are the same node). An open walk is where a and b are not the same node.

- *path*: a walk where no node is repeated
- *cycle*: a closed walk in which no other nodes are used more than once; other than the start and end nodes
- Euler circuit: a closed walk that uses each edge in the graph exactly once (no restriction on nodes)

connected: a graph is connected iff there is a walk between every pair of nodes
distance: between 2 nodes is the length of the shortest path between them
diameter: of a graph is the largest of all distances between pairs of nodes

Isomorphism

Intuition: two graphs are isomorphic if they have the same structure.



The above graphs are all isomorphic, we can disentangle them so they all appear in a straight line.

Suppose G_1 and G_2 are graphs where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. An *isomorphism* from G_1 to G_2 is a bijection $f : V_1 \to V_2$ such that edge (a, b) exists in G_1 iff edge $(f(a), f(b))$ exists in G_2.

To prove that two graphs are isomorphic, we should define that isomorphism (bijection).

There are many ways to show two graphs are *not* isomorphic:

- different numbers of nodes
- different numbers of edges
- structural differences: *e.g.,* number of nodes with degree k
- different subgraphs

subgraphs: $G' = (V', E')$ is a subgraph of $G = (V, E)$ iff $V' \subseteq V$ and $E' \subseteq E$, and the endpoints of E' are in V'.

So, if G_1 has a K_4 as a subgraph and G_2 does not, they are not isomorphic. You should let the reader know where the K_4 is (nodes a, b, c, d).