Tuesday 7/1: Functions

Functions Overview

Assume A and B are sets. Function f from A to B is an assignment of exactly one element of B to each element of A. We can write this function signature: $f : A \to B$. We call A the domain and B the co-domain (not *range* in this course, which you may have heard before; we use range in other ways)

Function example: Let $A = \{\text{Naina, Hongxuan}\}$ and $B = \{\text{blue, orange, purple}\}$, and let's define a function $c : A \to B$. One way we can define a function is by assigning each element of the domain an explicit output in the co-domain. For example, $c(\text{Naina}) = \text{orange}$ and $c(\text{Hongxuan}) = \text{blue}$. This is a valid function because every element in the domain is assigned to exactly one element in the co-domain. An invalid function would be, for example, $c(\text{Naina}) = \text{orange}$, $c(\text{Naina}) = \text{purple}$, $c(\text{Hongxuan}) = \text{blue}$. This violates us having "exactly one" output. We also cannot have $c(\text{Naina}) = \text{undefined}$ or $c(\text{Naina}) = \text{pink}$.

Onto

image: the image of a function $f : A \to B$ is the set of values produced when f is applied to elements of A. In other words, it is the outputs we actually get.

e.g., the image of the function c is $\{\text{orange, blue}\}$

onto: a function is onto iff the image equals the co-domain. Or, for function $f : A \to B$,

$\forall y \in B, \exists x \in A$ such that $f(x) = y$

e.g., the function c is not onto, because the image is $\{\text{orange, blue}\}$ but the co-domain is $\{\text{orange, blue, purple}\}$

a quick aside about nested quantifiers: ***order matters***

$\exists x \in A$ such that $\forall y \in B$, $f(x) = y$ means something completely different: we can find an element in the domain such that that element outputs *every* element in the co-domain; this is only possible if the co-domain has a cardinality of 1.

Easy onto example: Define a function $g : \mathbb{Z} \to \mathbb{Z}$, $g(x) = x + 2$. Prove g is onto.

Proof technique: We must show that for any arbitrary element y in the co-domain, it has a *pre-image* in the domain. In other words, we must find an $a \in \mathbb{Z}$ such that $g(a) = y$ for our *arbitrary* y.

Proof: Let $y \in \mathbb{Z}$ be an arbitrary element of the co-domain.

Then, let $a = y - 2$. a must be an element of the domain because y is an integer.

What does $g(a)$ evaluate to? $g(a) = y + 2 - 2 = y$.

Thus, we have found a pre-image for our arbitrarily chosen y, so g is onto.

note: finding the pre-image is usually more complicated, and will probably require more work than what is shown here. One technique is to find the "inverse" of function g on the side.

One-to-one

pre-image: for $f : A \to B$, if $y \in B$ and $x \in A$ and $f(x) = y$, the x is a pre-image of y (there could be more than one pre-image)

one-to-one: a function is one-to-one iff no element of the co-domain has more than one pre-image. In other words, every element of the domain should produce a unique output. Or, for function $f : A \to B$,

$\forall x, y \in A,\ x \neq y \rightarrow f(x) \neq f(y)$

However, using this definition directly is difficult in proofs, so we use the contrapositive:

$\forall x, y \in A,\ f(x) = f(y) \rightarrow x = y$

Concrete one-to-one example: Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ and let $f(x) = 2x + 1$. Prove f is one-to-one.

Proof technique: Assume two outputs are equal and show the inputs must have been equal as well.

Proof: Let $x, y \in \mathbb{Z}$ and suppose $f(x) = f(y)$.

Then, $2x + 1 = 2y + 1$

$2x = 2y$

$x = y$

So, f is one-to-one.

Bijection

bijection: if a function $f : A \rightarrow B$ is one-to-one and onto, it is a bijection. This means:

- $f^{-1} : B \rightarrow A$ exists and is also a bijection
- $|A| = |B|$

Function Composition

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = 3x + 7$ and $g : \mathbb{Z} \rightarrow \mathbb{Z}$, $g(x) = x - 7$.

Then, $(f \circ g)(x) = f(g(x)) = f(x - 8) = 3(x - 8) + 7 = 3x - 24 + 7 = 3x - 17$

See the textbook for one-to-one and onto proof examples that utilize composition; you'll have to know these!