Thursday 7/3: 2-Way Bounding

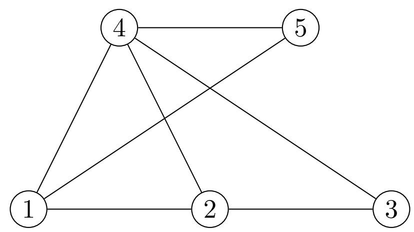
Sometimes it is difficult for us to prove a claim directly. Instead, it may be simpler to prove two weaker claims that together give us the claim we want.

For example, say we want to prove a claim that in some context, $x = 5$. If this is difficult, we might write one proof to show $x \geq 5$, and another proof to show $x \leq 5$. If both are true at the same time, it must be the case that $x = 5$. However, if we take this route, we *must* prove both claims; one is not sufficient.

Graph Coloring

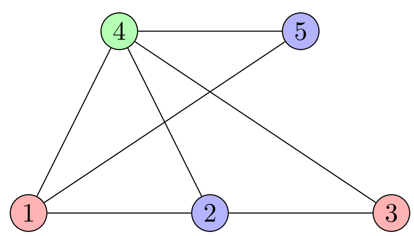
The *coloring* of a graph G assigns a coloring to each **node** in G such that no two *adjacent* nodes (ones that share an edge) have the same color.
chromatic number: of a graph G is the smallest number of colors needed to color G.

Chromatic number example: Given the below graph, what is the chromatic number?



The answer is **3**. How can we prove this? We have to give 2 arguments: an upper bound and a lower bound argument.

Upper bound: to prove the upper bound we just have to give an explicit coloring. On an exam you can write this out in words.



However, this only tells us that we need *at most* 3 colors. How do we know we can't complete the coloring with 2 colors? That's why we need the lower bound argument.

Lower bound: Since nodes 1,4,5 exist in a triangle (K_3), we know this structure requires *at least* 3 colors. Thus the entire graph also needs at least 3 colors. Note that you should know how many colors other structures require as well.

You may attempt to do both arguments at once, by giving a careful coloring, and showing that the way you colored the graph is the ***only*** *possible way. However, this is difficult to do correctly and we do not recommend it.*

Important note: the upper bound and lower bound argument have to be the *same* number for it to be a valid proof that this is the chromatic number.

Set Equality

We can show two sets are equal ($A = B$) by proving that $A \subseteq B$ and $B \subseteq A$.

Set equality example: Let $A = \{15p + 9q | p, q \in \mathbb{Z}\}$ and let $B = \{3k | k \in \mathbb{Z}\}$. Show $A = B$.

First let's show $A \subseteq B$.
Let $x \in \mathbb{Z}$ such that $x \in A$.
Then $x = 15p + 9q$ where $p, q \in \mathbb{Z}$.
So, $x = 3(5p + 3q)$.
Since $p, q \in \mathbb{Z}$, $5p + 3q \in \mathbb{Z}$, let's call this value r.
Then, $x = 3r$, so $x \in B$. So $A \subseteq B$.

Now let's show $B \subseteq A$.
Let $y \in \mathbb{Z}$ such that $y \in B$.
Then, $y = 3k$ for $k \in \mathbb{Z}$.
We can rewrite this as $y = -15k + 18k = 15(-k) + 9(2k)$.
We know $-k \in \mathbb{Z}$, let's call this a.
We know $2k \in \mathbb{Z}$, let's call this b.
Then, $y = 15a + 9b$ where $a, b \in \mathbb{Z}$, so $y \in A$. Thus $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, $A = B$.