Wednesday 6/25: Sets

Set Basics

A set, by definition, is an unordered collection of objects. The objects in a set (as we call *elements* or *members*) can be any type you want, such as numbers, colors, letters, or even sets.

A few things to note:

- Unordered-ness means that the order of elements in a set representation does not matter. For example, $\{1, 2, 3\}$ and $\{1, 3, 2\}$ are two names for the same set.
- Each element occurs only in a set, so $\{1, 2, 3\}$ and $\{1, 1, 2, 3, 3, 3\}$ are also two names of the same set.
- Sets are containers, so sets containing exactly one element are not the same as the one element itself. For example, $\{1\} \neq 1$.
- We use $\emptyset$ to denote an empty set. Please do not write $\{\}$, as that might make others think less of your mathematical skills. Note that $\{\emptyset\}$ denotes the set containing one element, which is the empty set.

You should have seen this notation $x \in A$, and it means x is a member of the set A.

Three basic ways to define a set:

- Listing all its elements, e.g. $\{0, 1, 2, 3\}$. If it is hard to list out all elements, please use the following two methods to avoid ambiguity, unless the pattern is VERY clear to your readers.
- Plain, mathematical English, e.g. "all integers greater than or equal to zero".
- Set builder notation, e.g. $\{x \in \mathbb{Z} | x \geq 0\}$.

The set builder consists of two parts separated by | or :. On the left we have two things: name (in this case x) and a range ($\mathbb{Z}$). On the right, we have more constraints that x must satisfy. Here is how to transform between verbal description and set builder notation, with corresponding parts using the same color.

The set of all integer x such that x is greater than or equal to 0 $\equiv \{x \in \mathbb{Z} | x \geq 0\}$

Cardinality of Sets

The cardinality of a (finite) set is defined by the number of *unique* elements in the set. An empty set by definition has cardinality of 0. Notation: $|A|$ means the cardinality of set A. We will talk about cardinality of infinite sets later in this course. Spoiler alert: there are different types of infinity!

A few examples:

- $|\emptyset| = 0$
- $|\{0\}| = 1$
- $|\{1, 2\}| = 2$
- $|\{1, 2, 2, 1\}| = 2$

Subsets

Given two sets A and B, A is a subset of B (written as $A \subseteq B$) if and only if every element in A is also in B. In other words, $\forall x, x \in A \implies x \in B$.

The bar at the bottom of $\subseteq$ suggests these two sets could be equal (similar to $\leq$). If you want to force two sets to be strictly different, meaning every element in A is also in B, but there is at least one element in B that is not in A, then you can say A is a *proper* subset of B, or $A \subset B$.

To prove $A \subseteq B$, use the following structure:

- pick arbitrary element $x \in A$
- show that $x \in B$

Note that $\emptyset \subseteq A$ for every set A. This statement can be translated into the following:

$$\forall x, x \in \emptyset \implies x \in A$$

Because the if condition is always false (we indeed cannot pick any element from an empty set), the whole statement is *vacuously true*. As a result, we claim that the empty set is a subset for every set, and it is a proper subset for every set other than empty set itself.

Note the difference between $\emptyset$ and $\{\emptyset\}$. $\emptyset$ is the empty set and has cardinality of 0. $\{\emptyset\}$ is a set containing one element, which is the empty set, and has cardinality of 1. $\emptyset \in \{\emptyset\}$.

Set Equality

To prove that two sets A and B are equal, you need to show the following:

- $A \subseteq B$
- and $B \subseteq A$

Set Operations

- Intersection ($\cap$): $A \cap B = \{x | x \in A \text{ and } x \in B\}$
- Union ($\cup$): $A \cup B = \{x | x \in A \text{ or } x \in B\}$
- Difference ($-$): $A - B = \{x | x \in A \text{ and } x \notin B\}$
- Cross Product ($\times$): $A \times B = \{(x, y) | x \in A \text{ and } y \in B\}$

For example, if we have $A = \{0, 1, 2\}$, $B = \{1, 3\}$, then

- $A \cap B = \{1\}$
- $A \cup B = \{0, 1, 2, 3\}$
- $A - B = \{0, 2\}$
- $A \times B = \{(0, 1), (0, 3), (1, 1), (1, 3), (2, 1), (2, 3)\}$