Thursday 6/26: Collections of Sets Pt.1

Collection

Collections are sets whose elements are other sets. Examples include $\{\emptyset\}$, $\{\{0\},\{1\},\{2\}\}$, $\{\{\{1\}\}\}$. Note that $\emptyset$ itself is also a collection, even if it contains nothing. In other words, a set A is a collection if $\forall x \in A, x$ is a set.

Powersets

If A is a set, the powerset of A (denoted as $\mathbb{P}(A)$) is the collection containing all subsets of A. In other words, $\mathbb{P}(A)$ contains all possible ways you can pick a subset from A. For example, for $A = \{1, 2, 3\}$, we have

$$\mathbb{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

A few observations:

- For any set A, $\mathbb{P}(A)$ must contain the empty set and A itself. Recall that $\emptyset \subseteq A$ and $A \subseteq A$ for any set A.
- $\mathbb{P}(\emptyset) = \{\emptyset\}$
- For any finite set A with $|A| = n$, the cardinality of its powerset, $|\mathbb{P}(A)|$, equals 2^n. Recall that the cardinality of powerset $\mathbb{P}(\mathbb{A})$ denotes how many different ways you can pick a subset from A. For each element you have two choices, to pick or not to pick. With n elements, that gives you 2^n total options.

Partitions

When we divide a base set A into non-overlapping subsets covering every element of A, we call the resulting collection a *partition* of A. The following rules must hold for a collection ($\{A_1, A_2, A_3...A_n\}$) to be called a partition of A:

- covers all of A: $A_1 \cup A_2 \cup A_3 ... \cup A_n$ = A
- non-empty: $A_i \neq \emptyset$ for all i
- no overlap: $A_i \cap A_j = \emptyset$ for all $i \neq j$

For example, given $A = \{1, 2, 3, 4, 5\}$, then $\{\{1, 2\}, \{3\}, \{4, 5\}\}$ is a partition of A, so is $\{\{1, 4, 5\}, \{3, 2\}\}$. In contrast, the following examples are not partitions of A:

- $\{\{1, 2\}, \{3, 6\}, \{4, 5\}\}$, because $6 \notin A$. This collection, however, is a partition of $\{1, 2, 3, 4, 5, 6\}$.
- $\{\{1, 2\}, \{3, 4\}, \{4, 5\}\}$, because $\{3, 4\}$, $\{4, 5\}$ have overlaps.
- $\{\{1, 2\}, \{3\}, \{4, 5\}, \emptyset\}$, because $\emptyset$ should not be included in a partition.

Trivially, $\{A\}$ is a partition of A for any set A (although it is not really partitioning A at all...).