Friday 6/27: Collections of Sets Pt. 2

Permutations

Suppose you have a set of n elements and you want to pick an ordered list of k elements from this set without repetition. How many ways can you pick this list? You have n choices for the first in the list, $n-1$ remaining choices for the second, $n-2$ choices for the third, and so on. In general, an ordered choice of k objects from a set of n objects is known as a k-permutation of the n objects. There are $n(n-1)...(n-k+1) = \frac{n!}{(n-k)!}$ different k-permutations of n objects. This number is called $P(n,k)$ (pronounced as "P of n, k", or "n permute k").

Permutations with Repetition

To count the number of permutations where there are repeating objects, we should first calculate the number of permutations as if everything is different, then divide by the number of permutations for each repeating part.

For example, suppose we are to count the number of ways to rearrange the string "COLLEGE". Note there are two L's and two E's.

The first step is to calculate the number of permutation for the seven letters as if the two L's are different (such as L_1 and L_2) and so are the two E's (E_1 and E_2). Therefore, we would have $\frac{7!}{(7-7)!} = 7!$ different permutations.

The second step is to deal with the repeating elements. For the two L's, there are 2! ways to arrange them, and ditto for the two E's.

Therefore, the final result should be $\frac{7!}{2! * 2!}$.

Combinations

In many applications, we have an n-element set and need to count all subsets of a particular size k. A subset of size k is called a k-combination. $C(n,k)$ (pronounced as "C of n, k", or "n choose k") (also written as $\binom{n}{k}$) represents the number of ways to choose k objects from n objects without caring the ordering (i.e. choosing subsets). This number is also called a "binomial coefficient".

To calculate the number $C(n,k)$, we also start by calculating $P(n,k)$, the number of ways to select an ordered list of k elements out of n elements. Then we need to count the number of different orders in which the same k-element list might appear, which is just $k!$. Therefore, the total number of combinations should be $\frac{P(n,k)}{k!} = \frac{n!}{(n-k)! * k!}$.

Note that:

- $\binom{n}{k}$ is only defined when $n \geq k \geq 0$.
- $\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{(n-k)! * k!}$.
- $\binom{n}{0} = 1$, meaning there is exactly one way to choose zero items from n items, which is to choose nothing.

Combinations with Repetition

Suppose I have a set S and I want to select a group of objects of the types listed in S, but I'm allowed to pick more than one of each type of object. For example, suppose I want to take 6 balls from a pile of unlimited red, blue, and yellow balls. I can take 2 blue balls and 4 yellow balls. Or I can pick 2 red, 1 blue, and 3 yellow.

Because ordering does not matter here, let's group the balls by colors. Between each pair of groups, we put a separator (e.g. #). Then, 2 blue 4 yellow would look like # B B # Y Y Y Y, and 2 red 1 blue 3 yellow would be R R # B # Y Y Y.

Suppose we fix that the first group must be red, second being blue, and third being yellow for example (but actually the ordering does not matter at all). Then we can simplify the representation without using the letters. 2 blue 4 yellow would just be # * * # * * * *, and 2 red 1 blue 3 yellow would be * * # * # * * *.

So now, we have transformed this problem into "out of 8 positions, how many ways are there to insert a separator?" In other words, $\binom{8}{2}$.

In general, suppose we are picking a group of k objects (with possible duplicates) from a list of n types. Then our picture will contain k stars and $n-1$ #'s. So we have $k+n-1$ positions in the picture and need to choose $n-1$ positions to contain the #'s. So the number of possible pictures is $\binom{k+n-1}{n-1}$.