Wednesday

Proof by contrapositive

Contrapositive: The rephrasing of an implication that is *equivalent* to the original claim.
e.g., the contrapositive of "if p then q" is "if not p then not q"
with quantifier: $\forall x,\ P(x) \to Q(x) \equiv \forall x,\ \neg Q(x) \to \neg P(x)$

importantly, the contrapositive is ***not*** *a negation*

and, we *only* do proof by contradiction on implications.

Easy contrapositive example: Use proof by contrapositive to prove the following claim: for any integers a, b if $a + b \geq 15$, then $a \geq 8$ or $b \geq 8$.

First, we state the contrapositive: for any integers a, b if $a < 8$ and $b < 8$ then $a + b < 15$.
Let a and b be integers where $a < 8$ and $a < 8$. Since a and b are integers, $a \leq 7$ and $b \leq 7$.
Then, $a + b \leq 14 < 15$. So $a + b < 15$.

Harder contrapositive example: Use proof by contrapositive to prove the following claim: for any integer n, if n^2 is even then n is even.

Contrapositive: For any integer n, if n is odd then n^2 is odd.
Let $n \in \mathbb{Z}$ and let n be odd.
By the definition of odd integers, $n = 2k + 1$ for some $k \in \mathbb{Z}$.
Then, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
Since $k \in \mathbb{Z}$, it is also the case that $2k^2 + 2k \in \mathbb{Z}$. So, by the definition of odd, n^2 is also odd.

You might be wondering why we ever do a proof by contrapositive. Usually it's because the direct proof is difficult; try proving this claim by direct proof for a challenge!