Tuesday

Predicate Logic

In predicate logic, we define functions that produce logical statements. For example, we might define $P(x)$ as "x is purple".

So if x is grape, then our statement is $P(\text{grape})$, which means grape is purple. This is true!

How can we use this predicate to write logical statements using quantifiers?

- $\exists x \in$ fruits such that $P(x)$. This means there is are one or more fruits that are purple. This is true.

  As an aside, you may see s.t. as an abbreviation for "such that".

- $\forall x \in$ fruits, $P(x)$. This means all fruits are purple. This is false.

- $\exists x, y \in$ fruits s.t. $P(x) \wedge \neg P(y)$. This means that there exists some purple fruit and some non-purple fruit. This is true.

  Variables x, y are the bound variables, and the scope of the statement is just the conjunction.

How can we negate statements with quantifiers? Let's define another predicate $Q(x)$ which means "x is sweet".

- $\neg(\forall x \in \text{fruits}, P(x)) \equiv \exists x \in \text{fruits s.t.} \neg P(x)$

- $\neg(\exists x \in \text{fruits s.t. } P(x) \wedge Q(x)) \equiv \forall x \in \text{fruits}, \neg(P(x) \wedge Q(x)) \equiv \forall x \in \text{fruits}, \neg P(x) \vee \neg Q(x)$

Proof Techniques

	Prove	**Disprove**
Universal ($\forall$)	direct proof	concrete counter-example
Existential ($\exists$)	concrete example	same as proving a universal—direct proof

- **Proving a universal statement:**

  For any integer k, if k is odd, then k^2 is odd. Prove this statement.

  Most importantly, you must assume the hypothesis is true and work toward the conclusion. DO NOT assume the conclusion is true.

  Let k be an odd integer. Then, $k = 2n + 1$ where $n \in \mathbb{Z}$.

  Then, $k^2 = (2n + 1)^2 = 4n^2 + 4n + 1$.

  This can be written as $2(2n^2 + 2n) + 1$. Since $n \in \mathbb{Z}$, Let's call integer $j = 2n^2 + 2n$.

  Then, $k^2 = 2j + 1$ where j is an integer. So, by the definition of odd integers, k^2 is odd.

  This is what we wanted to prove.

- **Proving an existential statement:**

  Prove that there exists an integer k such that $k^2 = 0$.

  $k = 0$ is such a $k \in \mathbb{Z}$: $0^2 = 0$.

- **Disproving a universal statement:**

  Every rational q has a multiplicative inverse. Disprove this.

  We define the multiplicative inverse r of q as $r \in \mathbb{Q}$ such that $rq = 1$.

  $q = 0$ does not have a multiple inverse, because $r \cdot 0 = 0 \neq 1$, regardless of our choice of r.

- **Disproving an existential statement:**

  There exists a $k \in \mathbb{Z}$ such that $k^2 + 2k + 1 < 0$. Disprove this.

  This is equivalent to proving: For all $k \in \mathbb{Z}$, $k^2 + 2k + 1 \geq 0$. And we prove this the same way we prove a universal, by direct proof. We leave this as an exercise for you.

Proof by cases example: When doing a direct proof, you might find that you need to split the hypothesis up into cases. Here is one example.

Prove the following using the definition of even:
For all $j, k \in \mathbb{Z}$, if j is even or k is even, then jk is even.

Case 1: Let $j, k \in \mathbb{Z}$ where j is even. Then $j = 2n$ where $n \in \mathbb{Z}$. Then, $jk = 2nk = 2(nk)$.
Since $n, k \in \mathbb{Z}$, $nk \in \mathbb{Z}$. So jk is even.
Case 2: is symmetric.