Discussion Problem Solutions for Examlet D

CS 173: Discrete Structures

Wednesday

Problem 1. from Invalid Recursion

- f is valid.
- g is invalid because neither case covers $n = 7$. (This also means that g is not defined for any larger value of n, e.g. $g(8)$ is undefined because its definition relies on $g(7)$.)
- h is invalid because $h(7)$ and above are undefined: $h(7)$ is defined in terms of $h(8)$, which is defined in terms of $h(9)$, etc in an *infinite* chain that never reaches a base case. (You could attempt to resolve this by reading this definition 'in reverse', i.e. if $h(n) = n + h(n+1)$ then $h(n+1) = h(n) - n$, which looks more like a valid definition. But notice that there is still no way to compute $h(7)$: you can't say $h(7) = h(6+1) = h(6) - 6$ because when $n = 6$ the "$n + h(n+1)$" case of h's definition does not apply.)
- s is invalid because both cases include $n = 7$ yet disagree on its value - does $s(7) = 2$ (from the first case), or does $s(7) = 7 + s(6) = 7 + 2 = 9$ (from the second case)?

(Note that all of the functions are technically well-defined if you restrict the domain far enough. For example, g is a well-defined function on the domain $\{6\}$, but that is not at all a "sensible" domain for a function whose definition claims to have an $n > 7$ case.)

Problem 12.2. in Discussion Manual

(b)
- $f(n) = 5f(n-1) + 1$
- $f(n) = 5(5f(n-2) + 1) + 1 = 5^2 f(n-2) + 5 + 1$
- $f(n) = 5(5(5f(n-3) + 1) + 1) + 1 = 5^3 f(n-3) + 5^2 + 5 + 1$

Based on the above, we predict the general form is that for any k,

$$f(n) = 5^k f(n-k) + \sum_{i=0}^{k-1} 5^i$$

The base case occurs when $n - k = 0$, i.e. when $k = n$:

$$f(n) = 5^n f(0) + \sum_{i=0}^{n-1} 5^i = \sum_{i=0}^{n-1} 5^i = \frac{5^n - 1}{4}$$

(c)
- $T(n) = 3T(\frac{n}{3}) + 13n$
- $T(n) = 3(3T(\frac{n}{3^2}) + 13\frac{n}{3}) + 13n = 3^2 T(\frac{n}{3^2}) + 13n + 13n$

- $T(n) = 3^2(3T(\frac{n}{3^3}) + 13\frac{n}{3^2}) + 13n + 13n = 3^3T(\frac{n}{3^3}) + 13n + 13n + 13n$

Based on the above, we predict the general form is that for any k,

$$T(n) = 3^k T(\frac{n}{3^k}) + k \cdot 13n$$

The base case occurs when $\frac{n}{3^k} = 1$, i.e. when $k = \log_3(n)$:

$$T(n) = 3^{\log_3(n)} T(1) + \log_3(n) \cdot 13n = 47n + 13n \log_3(n)$$

Friday

Problem 12.1. in Discussion Manual

(b) The first few values are:

$g(0) = 0 = \frac{3-0-3}{4}$

$g(1) = 1 + 3(0) = 1 = \frac{9-2-3}{4}$

$g(2) = 2 + 3(1) = 5 = \frac{27-4-3}{4}$

$g(3) = 3 + 3(5) = 18 = \frac{81-6-3}{4}$

Proof that $g(n) = \frac{3^{n+1}-2n-3}{4}$ by induction on n:

Base: $g(0) = \frac{3-0-3}{4} = 0$ ✓

Induction: Suppose (as our Inductive Hypothesis) that for any $n = 1...k-1$, $g(n) = \frac{3^{n+1}-2n-3}{4}$. Then, our goal is to show that $g(k) = \frac{3^{k+1}-2k-3}{4}$. We know by the definition of g that $g(k) = k + 3g(k-1)$. Since $n = k-1$ is covered by the IH, we know $g(k) = k + 3(\frac{3^k-2(k-1)-3}{4}) = k + 3(\frac{3^k-2k-1}{4}) = k + \frac{3*3^k-6k-3}{4}$. We can combine this into one fraction as $g(k) = \frac{4k+3*3^k-6k-3}{4} = \frac{3^{k+1}-2k-3}{4}$, which is what we wanted to show.

(d) The first few values are:

$x_1 = 1$

$x_2 = 7$

$x_3 = 7x_2 - 12x_1 = 7 \cdot 7 - 12 \cdot 1 = 37$

$x_4 = 7x_3 - 12x_2 = 7 \cdot 37 - 12 \cdot 7 = 175$

Proof that $x_n = 4^n - 3^n$ by induction on n:

Base: $x_1 = 1 = 4^1 - 3^1$ and $x_2 = 7 = 4^2 - 3^2$ ✓

Induction: Suppose (as our Inductive Hypothesis) that for any positive $i < k$, $x_i = 4^i - 3^i$. We know by the definition of the sequence that $x_k = 7x_{k-1} - 12x_{k-2}$. Since $k > 2$, $k-1$ and $k-2$ are both positive so we can apply the inductive hypothesis to get $7x_{k-1} - 12x_{k-2} = 7(4^{k-1} - 3^{k-1}) - 12(4^{k-2} - 3^{k-2})$. Finally, we simplify the right hand side as follows: $7(4^{k-1} - 3^{k-1}) - 12(4^{k-2} - 3^{k-2}) = (7 \cdot 4^{k-1} - 7 \cdot 3^{k-1}) - (3 \cdot 4^{k-1} - 4 \cdot 3^{k-1}) = 4^k - 3^k$. So $x_k = 4^k - 3^k$.

Monday

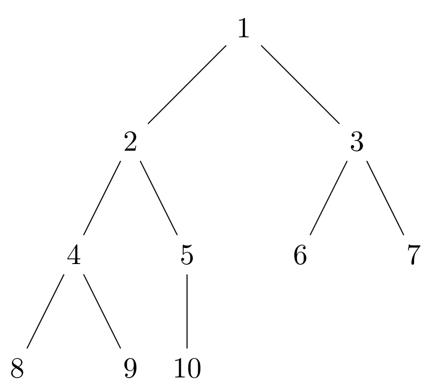
Problem 1. from Trees and Grammars

(a) See the diagram below for the tallest possible tree with 10 nodes.

(i) This tree has height 9.

(ii) The number of internal nodes is 9, since internal nodes include all nodes with at least one child node. That leaves one leaf node.

(b) See the diagram below for the shortest possible tree with 10 nodes.



(i) The height of this tree is 3.

(ii) This tree has 5 internal nodes and 5 leaves.

(c) No you cannot. A full binary tree requires that each node has either 0 or 2 children. In the diagram above, nodes 1 through 4 have two children each, nodes 6 through 10 have zero children, but node 5 only has one child. There is nowhere to move node 10 to avoid having one node with a single child.

(d) The number of nodes n in a full binary tree must be of the form$n = 2m + 1$ for some $m \in \mathbb{N}$.

Problem 2. from Trees and Grammars

(a) The following grammar will generate all palindromes consisting of "a"s and "b"s with start symbol S and terminal symbols a and b:

$$S \to aSa \mid bSb \mid b \mid a \mid \epsilon$$

(Commentary: if you forget the $S \to a \mid b$ rules, the grammar will only generate even-length palindromes, and if you forget $S \to \epsilon$ rule, the grammar will only generate odd-length palindromes.)

(b) Here is one way to write the grammar:

$$S \to aSa \mid B \mid a$$
$$B \to bBb \mid b \mid \epsilon$$