Discussion Problem Solutions for Examlet C

CS 173: Discrete Structures

Wednesday

Problem 8.4. in Discussion Manual

- K_n: 1, unless $n = 1$, then 0. K_1 has just one vertex (which is at distance 0 from itself), so it has diameter 0. For any larger complete graph, any two distinct nodes are at distance 1 because there is an edge from every node to every other. The diameter is thus 1 (regardless of how large n is).
- C_n: $\lfloor \frac{n}{2} \rfloor$. For even n, the maximum distance is $\frac{n}{2}$, i.e. the distance between two nodes that are exactly opposite each other. For odd n, the maximum distance is still between nodes that are as close to opposite as possible, but those nodes aren't quite opposite and the two paths between them are of lengths $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. The distance between them, and hence the diameter of the graph, is the smaller of the two, $\lfloor \frac{n}{2} \rfloor$. Thus in both even and odd cases, the diameter can be expressed as $\lfloor \frac{n}{2} \rfloor$. *There are other ways to express this answer, including:*

$$\begin{cases} n/2 & \text{when } n \text{ is even} \\ (n-1)/2 & \text{when } n \text{ is odd} \end{cases}$$

- W_n: 1 if $n = 3$, 2 otherwise. For $n = 3$, W_n is just K_4 which as we saw above has diameter 1. For any larger n, there are non-adjacent nodes in the rim so the diameter must be larger than 1, but there is a short path from any node to any other that goes through the 'hub' of the wheel, so the diameter is 2.

Problem 8.5. in Discussion Manual

1. One possible circuit is $ablefijmcdkgh$.
2. No Euler circuit is possible because there is at least one node (S) with odd degree. *(Note that there* does *exist an "Euler walk" - it is possible to start from S and end at the one other odd-degree vertex. But an Euler* circuit *must start and end on the same node.)*

Problem 9.1. in Discussion Manual

(b) No isomorphism is possible: in B_1 there are two vertices of degree 3 (B and D) and they are not adjacent, while in B_2 there are also two degree-3 vertices (3 and 6) but they *are* adjacent.

(There are other features you could use to prove non-isomorphism. For example, in B_2 the three nodes of degree 4 $(1, 4, 5)$ are all adjacent to each other; B_1 also has three nodes of degree 4 (F, E, A) but there is no edge between A and F.)

Thursday

Problem 10.1. in Discussion Manual

(b) We will proceed by proving that each of the two sets is a subset of the other.

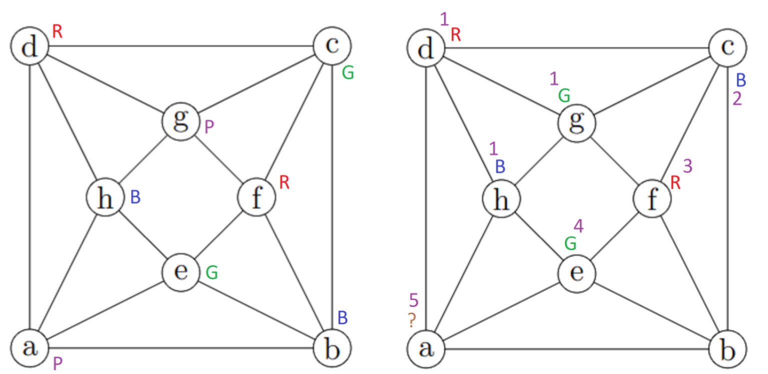
Subclaim: $X \subseteq Y$. Proof: Let z be an element of X. Then by definition of X, $z = 10x + 15y$ for some integers x, y. Factoring out the 5, we get $z = 5(2x + 3y)$. $2x + 3y$ is an integer since x, y are integers, so $z \in Y$.

Subclaim: $Y \subseteq X$. Proof: Let w be an element of Y. Then $w = 5k$ for some integer k. Then notice that $w = 10(-k) + 15k$. Since k is an integer, $-k$ is also an integer, so we see $w \in X$.

Since each set is a subset of the other, the two sets are equal.

Problem 10.2. in Discussion Manual

(d) Claim: the chromatic number $\chi(D)$ is 4. Proof: Figure 1a provides an upper bound of 4 by showing an explicit four-coloring, so it remains to show that the graph cannot be colored with 3 colors. We prove this as follows (see Figure 1b for a visualization): (1) Any 3-coloring must assign different colors to d, g, h; without loss of generality we call those three colors Red, Green, and Blue, respectively. (2) c is adjacent to d (Red) and g (Green), so it must be Blue. (3) f is adjacent to c (Blue) and g (Green), so it must be Red. (4) e is adjacent to f (Red) and h (Blue), so it must be Green. (5) Finally, a is adjacent to nodes of all three colors (d, h, e), so there is no possible color for a.



(a) A four-coloring of the graph.

(b) There is no three-coloring. (numbers correspond to steps in the proof)

Figure 1: Problem 10.2d

Notice that it would not *be enough to just have argued that* one particular *attempt at coloring with three colors didn't work. Instead, we argued that* every *attempt at three-coloring would run into this problem. At the beginning we do assign colors of our choice to d,g,h, but because color names are interchangeable and those three do need to be different from each other in any coloring, we haven't really made a significant choice, hence the "without loss of generality". For contrast, we could* not *have started the proof by assigning d,c,b to three different colors, because then we would not have addressed any colorings that may exist where d and b are given the* same *color.*

This proof could also be formalized as a proof by contradiction.

Friday

Problem 11.1. in Discussion Manual

(a) Proof by induction on n.

Base: Let $n = 1$. Then $\sum_{n=1}^{1} i^2 = 1 = \frac{1(2)(3)}{6}$.

Induction: Suppose (as our Inductive Hypothesis) that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n = 1...k-1$. Then our goal is to show $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

$$\begin{aligned}
\sum_{i=1}^{k} i^2 &= \sum_{i=1}^{k-1} i^2 + (k^2) && \text{(pulling a term out of the summation)} \\
&= \frac{(k-1)(k)(2(k-1)+1)}{6} + k^2 && \text{(by the Inductive Hypothesis)} \\
&= \frac{(k-1)(k)(2k-1)}{6} + k^2 && \text{(this and remaining steps are just algebra)} \\
&= \frac{(k-1)(k)(2k-1)}{6} + \frac{6k^2}{6} \\
&= \frac{(k)(k-1)(2k-1)+6k^2}{6} \\
&= \frac{(2k^3-3k^2+k)+6k^2}{6} \\
&= \frac{2k^3+3k^2+k}{6} \\
&= \frac{k(2k^2+3k+1)}{6} \\
&= \frac{k(2k+1)(k+1)}{6}
\end{aligned}$$

(b) We proceed by induction on n.

Base: Let $n = 1$. Then $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{(1+1)}$.

Induction: Suppose (as our Inductive Hypothesis) that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for each $n \in \mathbb{Z}^+$ less than some positive integer r. Then our goal is to show $\sum_{k=1}^{r} \frac{1}{k(k+1)} = \frac{r}{r+1}$.

$$\begin{aligned}
\sum_{k=1}^{r} \frac{1}{k(k+1)} &= \sum_{k=1}^{r-1} \frac{1}{k(k+1)} + \frac{1}{r(r+1)} && \text{(pulling a term out of the summation)} \\
&= \frac{(r-1)}{(r-1)+1} + \frac{1}{r(r+1)} && \text{(by the Inductive Hypothesis)} \\
&= \frac{(r-1)}{r} + \frac{1}{r(r+1)} && \text{(this and remaining steps are just algebra)} \\
&= \frac{(r-1)(r+1)+1}{r(r+1)} \\
&= \frac{r^2}{r(r+1)} \\
&= \frac{r}{r+1}
\end{aligned}$$

(c) Proof by induction on n.

Base: Let $n = 0$. Then $(\sum_{i=0}^{0} i)^2 = 0 = \sum_{i=0}^{0} i^3$.

Induction: Fix k and suppose that $(\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3$ for $n = 0, 1, \cdots, k-1$. Then we get the following:

$$\begin{aligned}
(\sum_{i=0}^{k} i)^2 &= (\sum_{i=0}^{k-1} i + k)^2 \\
&= (\sum_{i=0}^{k-1} i)^2 + 2k(\sum_{i=0}^{k-1} i) + k^2 \\
&\overset{IH}{=} \sum_{i=0}^{k-1} i^3 + 2k(\sum_{i=0}^{k-1} i) + k^2 \\
&= \sum_{i=0}^{k-1} i^3 + 2k\frac{(k-1)k}{2} + k^2 && \text{(by the given hint)} \\
&= \sum_{i=0}^{k-1} i^3 + k^3 \\
&= \sum_{i=0}^{k} i^3
\end{aligned}$$

Problem 11.4. in Discussion Manual

(Commentary: Obviously the proof must be wrong since the claim it is proving is clearly false. While that is not enough to say where *the flaw in the proof is, it does give us a good place to check: $P(1)$ is true but $P(2)$ is false, so we should look at the inductive step and carefully audit its argument that $P(1) \to P(2)$.)*

The argument implicitly relies on the fact that S' and S'' are not disjoint. If the sets overlap by even one horse H_*, then the proof is correct that all horses in the union are the same color, since all the horses in S' are H_*'s color and so are the horses in S''. However, consider the argument in the inductive step when $k = 2$. In this case, $S' = \{H_2\}$ and $S'' = \{H_1\}$, which are disjoint. Thus while it is true that all the horses in S' are the same color and all the horses in S'' are the same color, it is wrong for the proof to claim from this that all the horses in the union must also be the same color.

Monday

Problem 11.2. in Discussion Manual

Fix $a, b \in \mathbb{Z}$ and $p \in \mathbb{Z}^+$. Now we need to show $\forall n \in \mathbb{Z}^+, P(n)$, where $P(n)$ is "if $a \equiv b \pmod{p}$ then $a^n \equiv b^n \pmod{p}$". We proceed by induction on n:

Base: We need to show $P(1)$, i.e. that if $a \equiv b \pmod{p}$ then $a^1 \equiv b^1 \pmod{p}$. This is clearly true since $a = a^1$ and $b = b^1$.

Induction: Fix z, and suppose (as our Inductive Hypothesis) that for any i with $1 \leq i < z$, $P(i)$ is true. Now we need to show $P(z)$ is true, i.e. we need to show that if $a \equiv b \pmod{p}$ then $a^z \equiv b^z \pmod{p}$.

So suppose (towards direct proof) that $a \equiv b \pmod{p}$. Using this fact along with $P(z-1)$ (which is true by the IH), we also know that $a^{z-1} \equiv b^{z-1} \pmod{p}$. Multiplying our two equivalences together gives us $a \cdot a^{z-1} \equiv b \cdot b^{z-1} \pmod{p}$. This in turn gives us $a^z \equiv b^z \pmod{p}$, QED.

(Commentary: notice that the original claim has four variables in it - a, b, n, p. It would be valid to attempt an induction proof using any of those four as the induction variable, but if you pick something other than n in this case you will discover that there is no good way to finish the proof. So as always, don't be afraid to switch tactics if your current path seems not to be working.)

Problem 14.1. in Discussion Manual

(a) Proof by induction on n.

Base: When $n = 8$, we have $64 > 56 + 1$. This is true.

Inductive Hypothesis: Suppose that $n^2 > 7n + 1$ for any $n = 1...k-1$. Now our goal is to show that $k^2 > 7k + 1$.

Inductive step: $k^2 = (k-1)^2 + 2k - 1$. By the IH, $(k-1)^2 + 2k - 1 > 7(k-1) + 1 + 2k - 1 = 7k - 7 + 1 + 2k - 1 = 9k - 7$. Since $k > 2$, $9k - 7 > 7k + 1$. Thus, $k^2 > 7k + 1$.