Discussion Problem Solutions for Examlet A

CS 173: Discrete Structures

Thursday

Problem 1. Constructing a concrete relation

R is the relation which relates a to b for each row in this table:

a	b
1	2
2	1
1	1
2	2

The above solution is actually the *only* relation that satisfies the given conditions. To construct it, we can apply the conditions one at a time as follows: First, we know that $1R2$. Then to make R symmetric, we have to also include $2R1$. Then to make R transitive, we have to include $1R1$ and $2R2$. (Remember that the definition of transitive does not require that x, y, z be distinct.) Finally we check over all the conditions again to confirm we are done.

You did not have to demonstrate that this is the only possible R, but notice that if you include $3R3$ then the relation would be reflexive and thus an equivalence relation, and if you make any other additional pair of elements related, then either you will not add enough to restore symmetry and transitivity, or you will discover that everything has to be related to everything else, which is again an equivalence relation.)

Problem 4.3. in Discussion Manual

(a) By the definition of $\sim$, $[(1,3)]$ contains every (c,d) pair where $1+d = 3+c$, i.e., where $d-c=2$. (For example, it contains $(0,2)$ and $(1001,1003)$.) Similarly, we see that $[(0,4)]$ contains all pairs where $d-c=4$, and $[(2,4)]$ contains all pairs where $d-c=2$ (i.e., it is the same as our earlier $[(1,3)]$). More generally, for each integer k there is one equivalence class $[(0,k)] = \{(c,d) \mid d-c=k, c \in \mathbb{Z}, d \in \mathbb{Z}\} = \{(c, c+k) \mid c \in \mathbb{Z}\}$.

We will now show that $\sim$ is an equivalence relation by showing it is reflexive, symmetric, and transitive:

- Claim: $\sim$ is reflexive. Consider integers x and y. $x+y=y+x$, so $(x,y) \sim (x,y)$.
- Claim: $\sim$ is symmetric. Consider integers w,x,y,z such that $(w,x) \neq (y,z)$ and $(w,x) \sim (y,z)$. Then by definition of $\sim$, $w+z=x+y$. This can be rewritten as $y+x=z+w$, so $(y,z) \sim (w,x)$.
- Claim: $\sim$ is transitive. Consider integers u,v,w,x,y,z such that $(u,v) \sim (w,x)$ and $(w,x) \sim (y,z)$. Then by definition of $\sim$, $u+x=v+w$ and $w+z=x+y$. Then, from the first equation, $x=v+w-u$. Plugging this into the second, we have $w+z=v+w-u+y$. This simplifies to $u+z=v+y$. Thus, $(u,v) \sim (y,z)$.

(b) By the definition of $\sim$, $[2]$ contains every integer y where $4 \mid (3 \cdot 2 + 5y)$. Since $6+5y$ and $y-2$ differ by a multiple of 4 (i.e. $4(y+2)$), we see that $4 \mid (6+5y)$ iff $4 \mid (y-2)$, so $[2]$ contains all integers that are 2 more than a multiple of 4, i.e. $[2]$ is precisely $[2]_4$ (the congruence class of 2 mod 4).

Similarly, $[3]$ contains every integer y where $4 \mid (3 \cdot 3 + 5y)$. Since $9+5y$ and $y-3$ differ by a multiple of 4 (i.e. $4(y-3)$), we see that $4 \mid (9+5y)$ iff $4 \mid (y-3)$, so $[3]$ contains all integers that are 3 more than a multiple of 4, i.e. $[3]$ is just $[3]_4$.

By continuing the same logic, we see that there are only 4 equivalence classes and these are the congruence classes of 0, 1, 2, and 3 mod 4. In fact $\sim$ *is* just equivalence mod 4, i.e. $x \sim y$ if and only if $x \equiv y \pmod 4$.

We will now show $\sim$ is an equivalence relation by showing it is reflexive, symmetric, and transitive:

- Claim: $\sim$ is reflexive. Consider an integer x. $3x + 5x = 8x = 4(2x)$, so $4 \mid 3x + 5x$ and thus $x \sim x$.
- Claim: $\sim$ is symmetric. Consider integers x, y such that $x \neq y$ and $x \sim y$. Then by definition of $\sim$, $4 \mid 3x + 5y$, i.e. $3x + 5y = 4k$ for some integer k. Then $5x + 3y = 8x + 8y - (3x + 5y) = 8x + 8y - 4k = 4(2x + 2y - k)$. Thus $3y + 5x$ is a multiple of 4, so $y \sim x$.
- Claim: $\sim$ is transitive. Consider integers x, y, z such that $x \sim y$ and $y \sim z$. Then by definition of $\sim$, $4 \mid 3x + 5y$, and $4 \mid 3y + 5z$. i.e. $3x + 5y = 4k$ and $3y + 5z = 4m$ for some integers k, m. Adding the equations gives us $3x + 8y + 5z = 4k + 4m$, so $3x + 5z = 4(k + m - 2y)$ and thus $x \sim z$.

Friday

Problem 5.1. in Discussion Manual

(b) This is true: 0 is a value for x such that, for every possible value of y, $xy = x$. *(The domain is very important here - if instead it were e.g. positive integers, then the statement would be false.)*

(c) This is true: every real number x has a decimal expansion. If we truncate this expansion to two decimal places, we will have a rational number y. Then, x and y will differ by at most 0.009999..., which is less than or equal to 0.01.

Problem 5.2. in Discussion Manual

(a) This translates to "for every element in the co-domain, there is an element of the domain which maps to it", which is just the definition of *onto*.

(b) This translates to "every element in the domain maps to (at least) one element of the co-domain". This statement is true of *all* functions, so it doesn't represent any interesting named concept.

(c) This translates to "there is an element of the co-domain which is mapped to from every element of the domain". In other words, this represents the concept of a function being *constant* - e.g. this statement is true of the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 42$.

(d) This translates to "there is an element of the domain which maps to *every* element of the co-domain". Since every input to a function must map to *exactly* one output, the statement is true if and only if the co-domain has exactly one element.

Problem 5.3. in Discussion Manual

(b) Let f be a function from $\mathbb{Z}^2$ to $\mathbb{Z}$, and let z be an arbitrary element in $\mathbb{Z}$ (the co-domain of f). We need to find a pre-image for z in $\mathbb{Z}^2$. Let's call that pre-image (a, b), and suppose $a = 1$ and $b = z - 27$. We know 1 is an integer, and $z - 27$ is also an integer since z and 27 are both integers. So, $(a, b) \in \mathbb{Z}^2$. Then, $f(a, b) = f(1, 27 - z) = 1 * (z - 27) + 27 = z$. Thus, we have found a pre-image to z, and f must be onto.

Monday

Problem 7.1. in Discussion Manual

(b) Let h be the given function and let x and y be elements of $\mathbb{N}$. Let's suppose $h(x) = h(y)$. Then, by the definition of h, $x^2 + 27 = y^2 + 27$. Simplifying, we get $x^2 = y^2$. Since x and y are both natural numbers, it then must be the case that $x = y$. Thus, we have proven that h is one-to-one.

Problem 7.3. in Discussion Manual

(a) Suppose that $f \circ g$ is onto and f is one-to-one. Let y be an element of B, and consider $z = f(y)$. Since $f \circ g$ is onto, there is an element $k \in A$ such that $(f \circ g)(k) = z$, i.e. $f(g(k)) = z$. Since f is one-to-one and $f(g(k)) = z = f(y)$, it follows that $g(k) = y$. Thus we have found an element of A which maps to y, so g is onto.

(b) Let $A = \{a\}$, $B = \{b, z\}$, $C = \{c\}$. Let f and g be the constant functions $f(x) = c$ and $g(x) = b$. Then $f \circ g$ is onto (because $f(g(a)) = c$), but g is not onto (because there is no input which gives the output z).