Recursion and Trees and Grammars Questions — CS 173 Staff

Last Edited on 7/6/24 at 14:17

Problem 1

Recall that the Fibonacci numbers $F_0, F_1, \ldots, F_n, \ldots$ are defined as follows for integers $n \geq 0$:

$$F_n = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ F_{n-2} + F_{n-1} & n \geq 2 \end{cases}$$

So the first few values are $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$.

Let $r = \frac{1+\sqrt{5}}{2}$ and notice that $r^2 = r + 1$. Prove the following claim using induction:

For all integers $n \geq 2$, $F_n \geq r^{n-2}$.

When stating your inductive hypothesis, be explicit in what the hypothesis says rather than just referring to “the claim” or “this.”

Problem 2

Consider a function f as follows defined for integers $n \geq 3$:

$$f(3) = 8$$
$$f(n) = 5f(n-1) \text{ if } n \geq 4$$

Find the closed form of f via unrolling.

Problem 3

Now *prove via induction* that the closed of f you just found in Problem 2 is indeed the closed form.

Problem 4

Recall the closed form for a geometric series. Let $r \neq 1 \in \mathbb{R}$ be a real number. Then

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}.$$

Consider a function f as follows defined for powers of 4:

$$f(1) = 0$$
$$f(n) = 2f(n/4) + n \text{ if } n \geq 4$$

Assume that n is a power of 4, so $n = 4^k$ for some $k \geq 0 \in \mathbb{Z}$. Find the closed form of f via unrolling.

Problem 5

(i) How many nodes are in a full and complete binary tree of height h? How many nodes are in a full and complete m-ary tree of height h?

(ii) What level is the root node of a binary tree of height h? What level is a leaf node in a binary tree of height h? Does this change if we have a ternary tree or a general m-ary tree?

(iii) Consider the following grammar G with start symbol S and terminals a and b:

$$S \rightarrow \varepsilon \mid a\ S\ a \mid b\ S\ b$$

Give 2 examples of strings generated by G, along with their parse trees, as well as 2 strings not generated by G. What type of strings does this grammar generate?