Discussion Problem

CS 173: Discrete Structures

Problem 1. Logical Reasoning

Suppose we are given the following facts:

1. All chestnut-eating animals are fun-loving
2. No penguin eats mulberries
3. Some well-dressed animals are uncomfortable
4. At least one penguin is uncomfortable
5. All animals eat mulberries or chestnuts
6. No uncomfortable animal eats mulberries

Which of the following statements can be proved (inferred) from the above facts?

You may assume that "not comfortable" is the same as "uncomfortable". You may also assume that penguins are known to be animals.

Select one or more:

(a) Every comfortable penguin eats mulberries

(b) At least one penguin is well-dressed

(c) No penguins are fun-loving

(d) There is at least one fun-loving well-dressed animal

(e) All penguins are fun-loving

(f) All penguins are uncomfortable

Notes:

Suggest students to draw an implication graph to keep track of the universal facts. Make sure they model both the implications and their contrapositives when forming the graph.

In the notes below, when something is not provable, we give a concrete counterexample (one counterexample works for all). A counterexample is a world where the facts hold but the given statement does *not* hold. Guide them towards finding concrete counterexamples once they finish proving what they can prove (i.e., (d) and (e)).

Some small things:

- When modeling (5), pause and note that the statement parses as $\forall x.\ m(x) \lor c(x)$ and not $\forall x. m(x) \lor \forall x. c(x)$. Ask students whether they are different (they are). Construct a world where they evaluate differently.
- (6) is a negated existential statement. Pause here and show how it converts to a more friendly universal statement.

Solution:

Let us first model the known facts using (first-order) logic. Since all penguins are animals, we can implicitly assume that whenever we talk about something, it's an animal (i.e., variables range over animals).

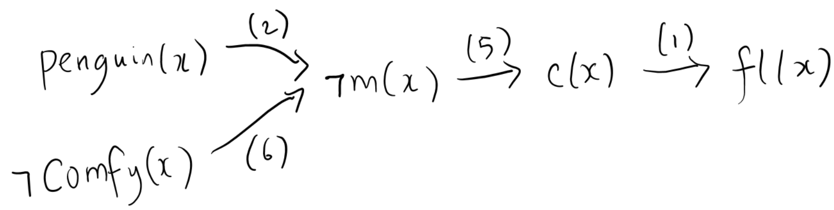
Let us use the following predicates to talk about animals:

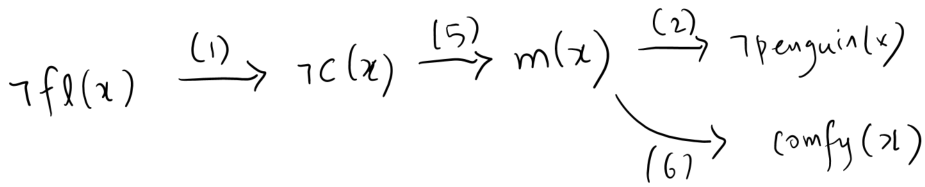
- $c(x)$: x is chestnut-eating
- $fl(x)$: x is fun-loving
- $penguin(x)$: x is a penguin
- $m(x)$: x eats mulberries
- $wd(x)$: x is well-dressed
- $comfy(x)$: x is comfortable

We can then formulate the facts as follows. Let us write them using *implications* as much as possible.

1. $\forall x.\ c(x) \Rightarrow fl(x)$
2. $\forall x.\ penguin(x) \Rightarrow (\neg m(x))$
3. $\exists x.\ wd(x) \land \neg comfy(x)$
4. $\exists x.\ penguin(x) \land \neg comfy(x)$
5. $\forall x.\ m(x) \lor c(x)$, which is equivalent to $\forall x.\ (\neg m(x)) \Rightarrow c(x)$
6. $\neg \exists x.\ (\neg comfy(x) \land m(x))$, which is equivalent to $\forall x.\ (comfy(x) \lor \neg m(x))$, which is equivalent to $\forall x.\ (m(x) \Rightarrow comfy(x))$.

To aid reasoning, let's draw a graph that captures *universal* implications (we will leave the existential formulas out). In this graph, for every formula of the form $\forall x.\alpha(x) \Rightarrow \beta(x)$, we will throw in an arrow from $\alpha(x)$ to $\beta(x)$. Let us also throw in the equivalent contrapositives as well— since $\forall x.\alpha(x) \Rightarrow \beta(x)$ is equivalent to $\forall x.(\neg(\beta(x))) \Rightarrow (\neg(\alpha(x)))$, we will throw in an arrow from $\neg\beta(x)$ to $\neg\alpha(x)$ as well.





Note that the above captures only the universal statements, i.e, (1), (2), (5), and (6). It does not capture the existential statements (3) and (4).

Let us now turn to the various statements.

(a) Every comfortable penguin eats mulberries.

From the above graph, it certainly doesn't seem that this statement is true. In fact, if x is a penguin that is also comfortable, the graph says it will definitely not eat mulberries.

Let us now construct a concrete counterexample to the statement. Assume there is a comfortable penguin, call it p, that does *not* eat mulberries. Let's see if we can build a world with p that is consistent with all known facts.

(2) says no penguin eats mulberries. This is fine for p as it does not eat mulberries. (5) is also consistent, but in order to satisfy (5), we must allow that p eats chestnuts. (6) is consistent with our assumptions, since p is anyway comfortable, and (6) talks about uncomfortable animals only. (1) talks about chestnut-eating animals, and so we must allow that p is fun-loving. We are consistent with (3) as (3) talks only about some well-dressed animals, and hence need not talk about p (i.e., we can assume p is not well-dressed). We can be consistent with (4) as (4) talks about some penguin, which need not be p.

We can in fact now construct a world W where there are just three animals (p, p', and w) that satisfies all the facts:

- p is a penguin who is comfortable and does not eat mulberries and eats chestnuts and is fun-loving. It's also not well-dressed.

  Formally, $penguin(p)$, $comfy(p)$, $\neg m(p)$, $c(p)$, $fl(p)$, $\neg wd(p)$ hold.
- There is a well-dressed animal w, that is a not a penguin, that is uncomfortable (to satisfy (3)). We can have this animal not eat mulberry (to satisfy (6)), and eat chestnuts (to satisfy (5)), and be fun-loving (to satisfy (1)).

  Formally, $\neg penguin(w)$, $wd(w)$, $\neg comfy(w)$ hold, $\neg m(w)$, $c(w)$, $fl(w)$ hold.

- There is another penguin p' (different from p) which is uncomfortable, does not eat mulberries, eats chestnuts, and is fun-loving (to satisfy (4)), and is not well-dressed. Formally, $penguin(p')$, $\neg comfy(p')$, $\neg m(p')$, $c(p')$, $fl(p')$, $\neg wd(p')$ hold.

It is easy to see that this world satisfies all the facts but does not satisfy the given statement (which is, formally, $\forall x.(penguin(x) \wedge comfy(x)) \Rightarrow m(x))$.

Hence the given statement does not logically follow from the facts.

(b) At least one penguin is well-dressed.

The only fact that allows for at least a penguin, well-dressed or otherwise, to exist is (4). That penguin is uncomfortable. But it needn't be well-dressed. In fact the world W we constructed for the subproblem (a) shows that there need not be a well-dressed penguin. So this statement is not entailed by the facts.

(c) No penguins are fun-loving.

Again, the world W we painted above shows that there can be fun-loving penguins. So the statement is not entailed by the facts.

(d) There is at least one fun-loving well-dressed animal.

This is provable. We know from (3) that there is a well-dressed animal that is also uncomfortable. From the graph, we see that uncomfortable animals must be fun-loving.

A more formula argument (following the implications in the graph) is as follows:

- From (3), we know there is at least one well-dressed animal that is also uncomfortable; call this animal a.
- From (6), we know that a does not eat mulberries.
- From (5), we know that p must eat chestnuts.
- From (1), we know that p must be fun-loving.
- Hence there is at least one fun-loving well-dressed animal.

(e) All penguins are fun-loving.

From the graph, it's clear that all penguins must be fun-loving.

A more formula argument (following the implications in the graph) is as follows:

- Let p be any penguin.
- From (2), we know that p does not eat mulberries.
- From (5), we know that p must eat chestnuts.
- From (1), we know that p must be fun-loving.

Note that the above argument is true even if there are no penguins. Of course, (4) says there is at least one penguin. But we didn't use (4) for the above argument.

(f) All penguins are uncomfortable.

This statement doesn't seem to hold looking at the graph as it looks like all penguins are in fact comfortable. In fact, the statement is false in the world W we created for (a) (where p is a comfortable penguin). So the statement is not entailed by the facts.