Sets

CS 173

Summer 2022

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Learning Objectives

- Know the definitions of basic set operations
- Be able to apply these operations to given sets
- Understand how the cardinality of a set is affected by these operations
- Prove that one set is a subset of another

What is a set?

Definition: A *set* is an unordered collection of objects

Examples:

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$
- $\{2, 3, 5, 7\}$
- {Illinois, California, New York}
- {Illinois, 3, $\mathbb{R}$}
- $\{\} = \emptyset$
- $\{x \in \mathbb{Z} \mid x \leq 10 \text{ and } x \text{ is prime}\}$
- $\{p/q \mid p, q \in \mathbb{Z}\}$
- The equivalence class of 4 mod 7:
    - $\{\dots -10,\ -3, 4, 11, 18, \dots\}$
    - $\{x \in \mathbb{Z} \ \mid x \equiv 4 \pmod 7\}$
    - $\{x \in \mathbb{Z} \ : 7 \mid (x - 4)\}$
    - $\{x + 4 : x \in \mathbb{Z} \text{ and } 7 \mid x\}$

Set Builder Notation

How could we write these sets in symbols?

- The set of all 3-tuples whose elements are in increasing order
- The set of all squares of multiples of 7

Cardinality

Definition: The *cardinality* of a set S, denoted $|S|$, is the number of distinct elements that it contains.

Examples:

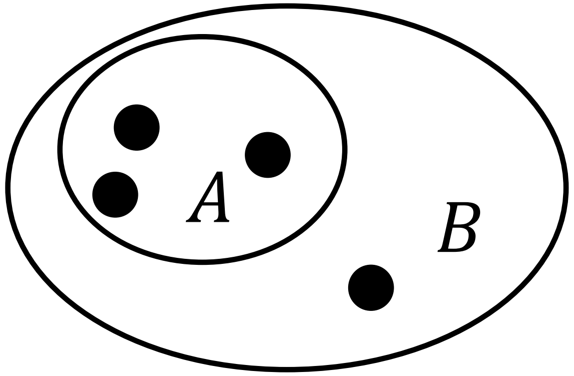
- $|\{2, 3, 5, 7\}| =$
- $|\mathbb{R}| =$
- $|\{\text{Illinois}, 3, \mathbb{R}\}| =$
- $|\emptyset| =$
- $|\{2, 3, 2\}| =$
- $|(1,2), (2,1)| =$

Subsets

Definition: Given two sets A and B we say that A is a *subset* of B (written $A \subseteq B$) if every element of A is also in B.
If $A \subseteq B$ and $A \neq B$ then we say that A is a *proper subset* of B (written $A \subset B$).

Examples:

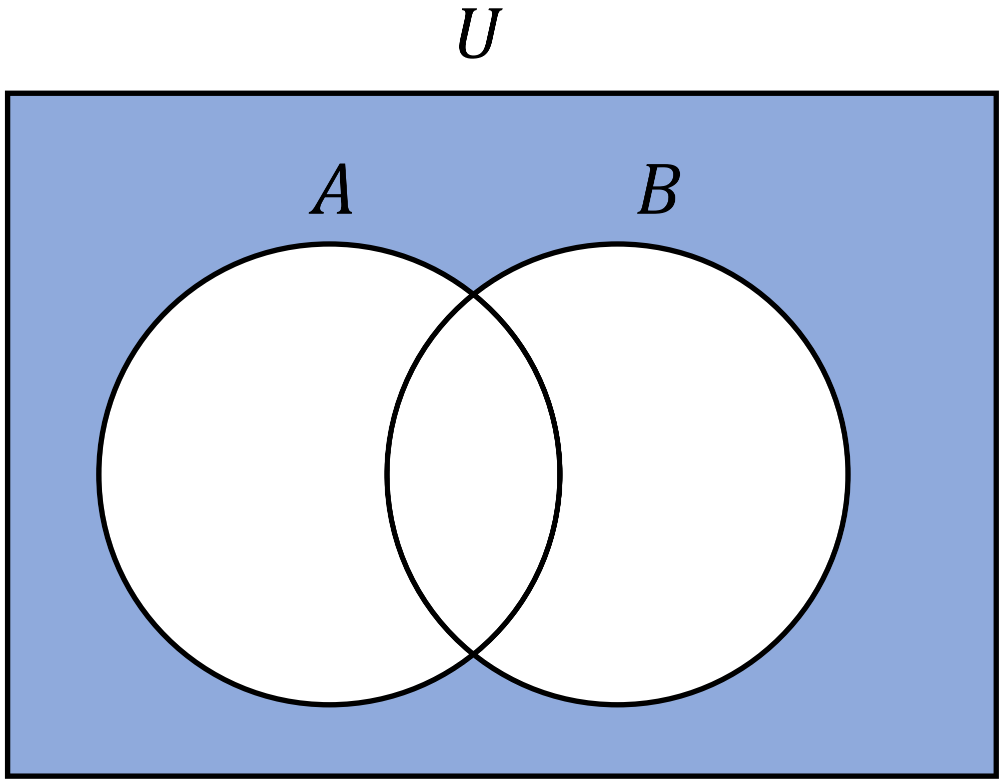
- $\{2,3\} \subseteq \{1,2,3\}$
- $\mathbb{Z} \subseteq \mathbb{Q}$
- $\mathbb{Z} \subseteq \mathbb{Z}$
- $\emptyset \subseteq \mathbb{Z}$



$A \subseteq B$

Set Operations

- Intersection
  - $A \cap B = \{x : x \in A \wedge x \in B\}$
- Union
  - $A \cup B = \{x : x \in A \vee x \in B\}$
- Difference
  - $A - B = A \setminus B = \{x : x \in A \wedge x \notin B\}$
- Complement
  - $\bar{A} = \{x \in U : x \notin A\} = U \setminus A$

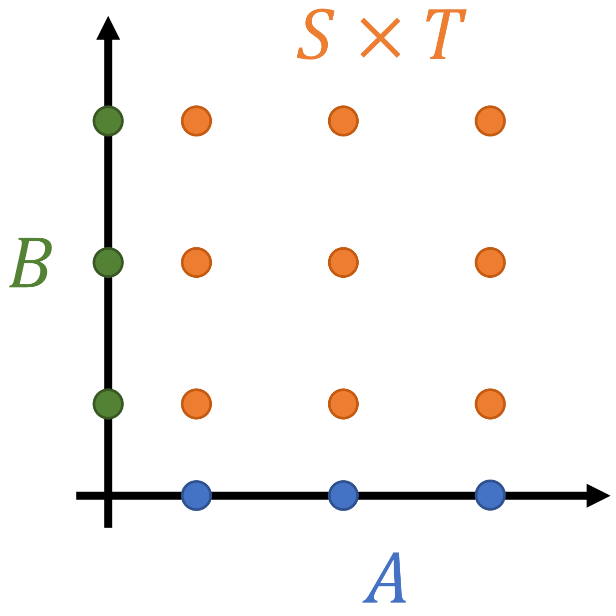


Cartesian Product

$$A \times B = \{(x, y): x \in A, y \in B\}$$

Examples:

- $\mathbb{R} \times \mathbb{R}$
- $\mathbb{Z} \times \mathbb{Z}$
- $\mathbb{R} \times \mathbb{Z}$
- $\{1, 2\} \times \{3, 4\} =$
- $\{3,4\} \times \{1,2\} =$
- $\emptyset \times \{1, 2\} =$
- $\{1\} \times \{2\} \times \{3,4\} =$



Practice with Set Operations

$A = \{x \in \mathbb{Z} : |x| \leq 5\}$
$B = \{3x : x \in \mathbb{N}\}$
$U = \mathbb{Z}$

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Some properties of Set Operations

- Commutative (∩,∪)
  - $A \cap B = B \cap A$
  - $A \cup B = B \cup A$
- Associative (∩,∪)
  - $A \cap (B \cap C) = (A \cap B) \cap C$
  - $A \cup (B \cup C) = (A \cup B) \cup C$
- Distributive (∩ over ∪, and ∪ over ∩)
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Some properties of Set Operations

- Double complement
  - $\overline{(\bar{A})} = A$
- De Morgan's Laws
  - $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$
  - $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

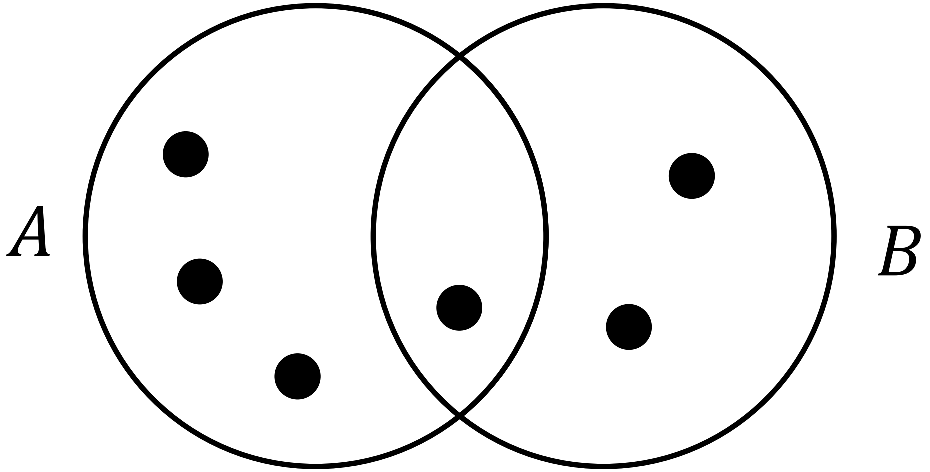
Cardinality of Cartesian Product

- $|A \times B| =$
- $|A \times B \times C| =$

Flavor:	**Sugar:**	**Toppings:**
Black	Regular	Boba
Green	Less	Grass Jelly
Oolong	None	Crystal Boba
Taro		Mango Popping Boba
		Pudding

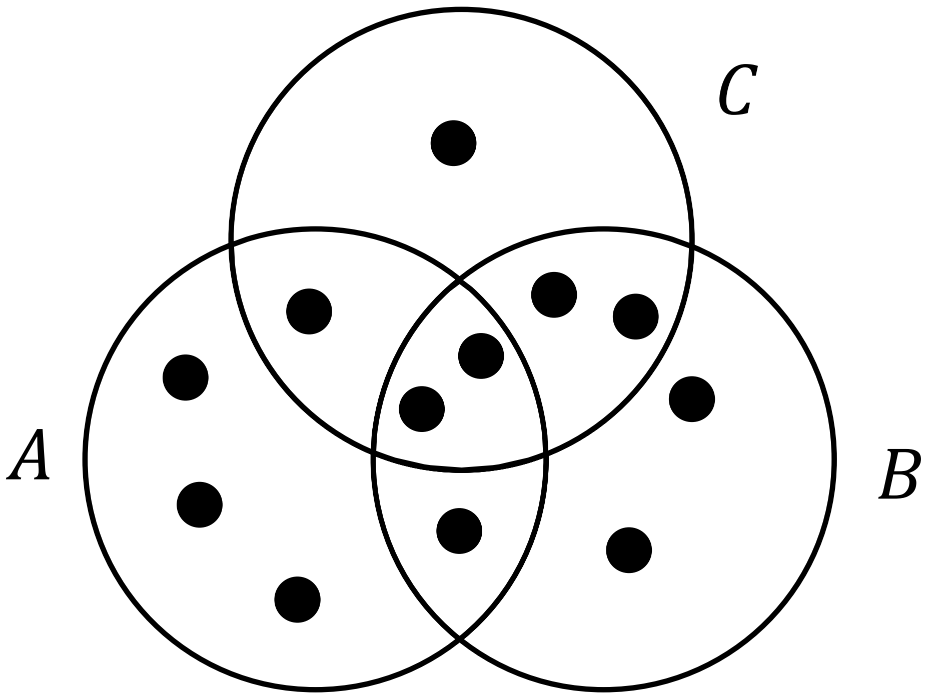
Inclusion-Exclusion Principle

$|A \cup B| =$



Inclusion-Exclusion Principle

$|A \cup B| = |A| + |B| - |A \cap B|$
$|A \cup B \cup C| =$



Using the Inclusion Exclusion Principle

<u>Example:</u> How many integers between 1 and 100 are divisible by 2 or 5?

Using the Inclusion Exclusion Principle

Example: A class has 30 students

- 15 own dogs
- 12 own cats
- 8 own birds
- 5 own a dog and a cat
- 4 own a dog and a bird
- 1 owns a cat and a bird
- Nobody owns a cat, a dog, and a bird.

How many students have no pets?

Proving Set Inclusion

$$A \subseteq B \Leftrightarrow \forall x \in A, x \in B$$

Proof Outline:

Let $x \in A$ be arbitrary.
[Show that $x \in B$]
So $x \in B$. Since x was arbitrarily chosen, we conclude that $A \subseteq B$.

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Example: Let $\mathrm{A} \coloneqq \{x \in \mathbb{Z} : x^2 + 1 \text{ is even}\}$ and $B \coloneqq \{y \in \mathbb{Z} : y \text{ is odd}\}$ Show that $A \subseteq B$.

Another Set Inclusion Proof

<u>Example:</u> Let A $:= \{(x, y) \in \mathbb{R}^2 : x > y > 5\}$, $B := \{(p, q) \in \mathbb{R}^2 : pq = 100\}$, and $C := \{(a, b) \in \mathbb{R}^2 : a < 20\}$ Show that $A \cap B \subseteq C$.

An Abstract Set Proof

Example: Let A, B, C be sets. Prove that $(B \setminus A) \cup C \subseteq (B \cup C) \setminus (A \setminus C)$.

Another Abstract Set Proof

Example: Let A_1, A_2, B_1, B_2 be sets. Prove that $(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$.

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Announcements

- Examlet 2 will be proctored through CBTF
  - Sign up via Prairetest
  - Sign up for a make up examlet if you missed examlet 1!
- We have a Discord now! (see Piazza for link)