Discrete Math

Partitions and State Diagrams

CS 173

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Learning Objectives

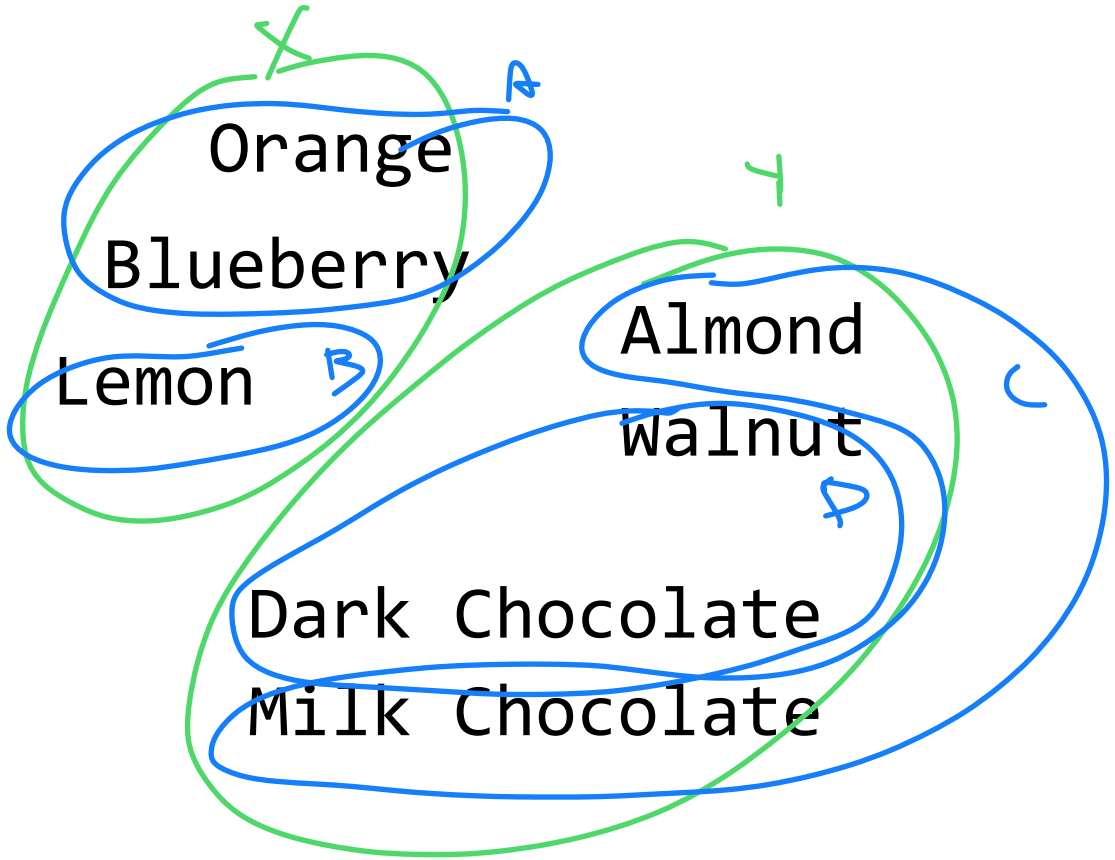
Review combinations in the context of partitions

Introduce the binomial theorem and review probabilities

Introduce state diagrams and practice many different forms

Partitions

A **partition** of A is a collection of non-empty subsets of A, such that each element of A is contained by exactly one subset.



Any combination of subsets!

$A \cup B \cup C \cup D$

$X \cup Y$

No overlap!

$X \cap Y = \emptyset$

Partitions

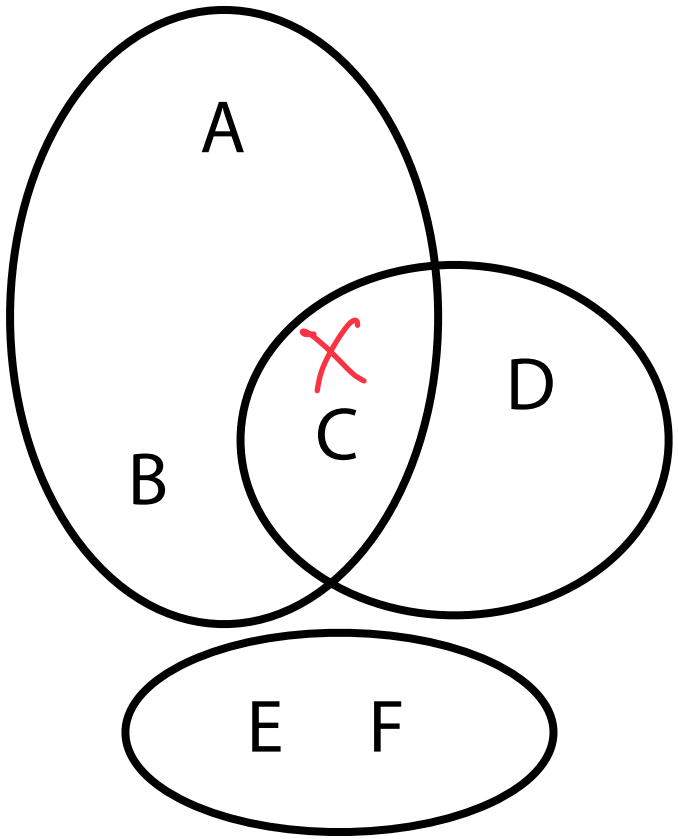
A collection of sets C is a partition of A iff

(1) C covers A: $C_1 \cup C_2 \cup \ldots \cup C_n = A$

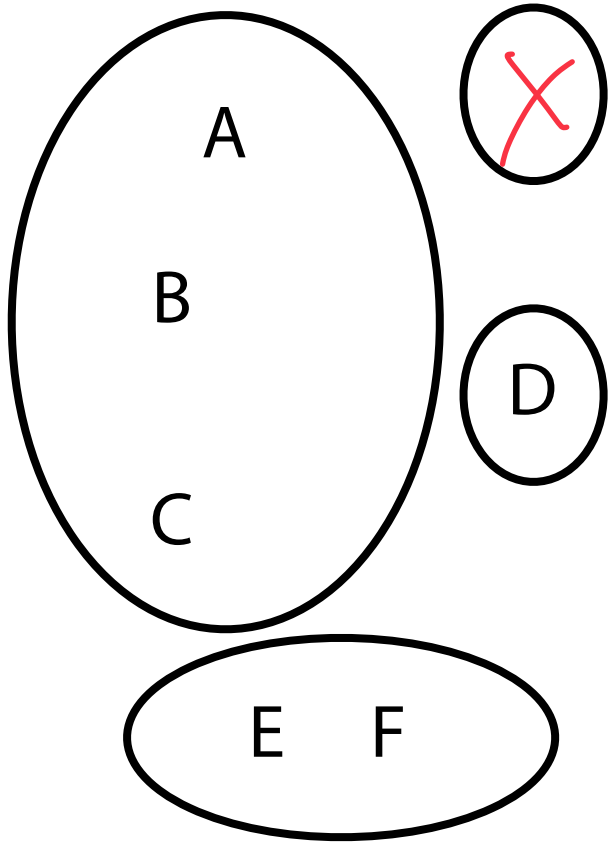
(2) No overlap within elements of C: $C_i \cap C_j = \emptyset$ (if $i \neq j$)

(3) No element of C is empty

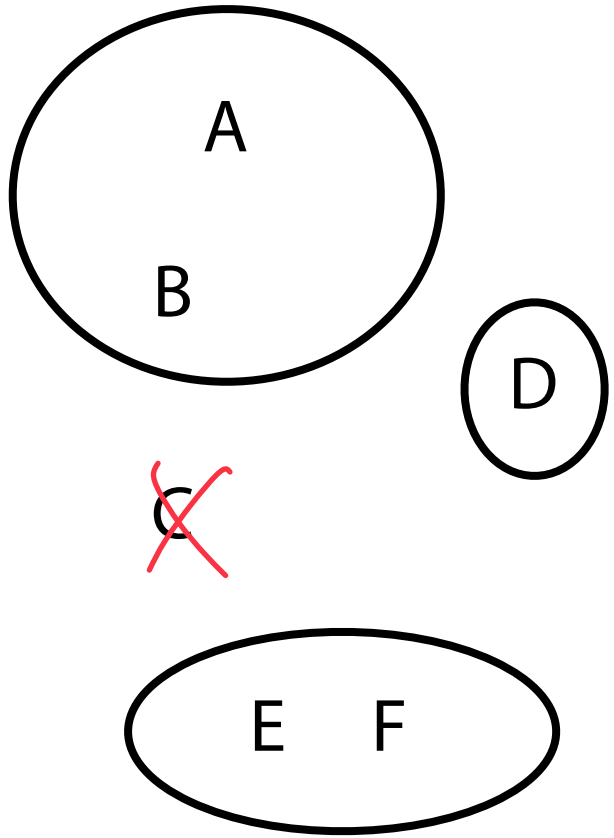
Partitions



No overlap

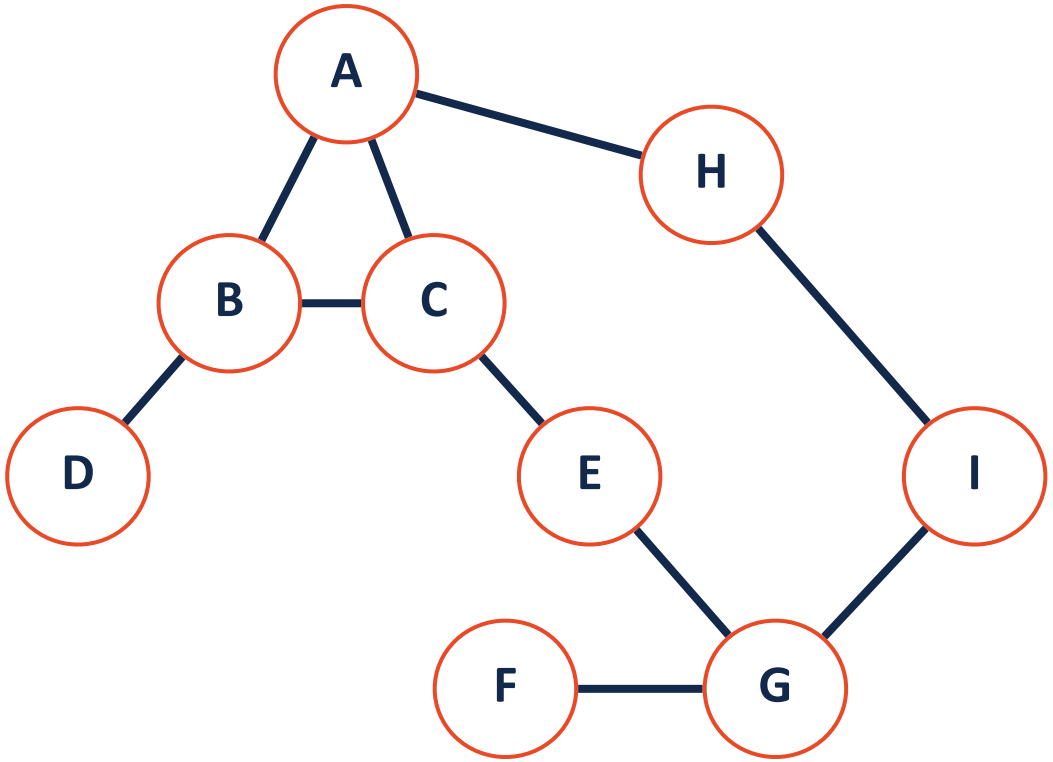


No empty sets



Must cover all objects

Partitions $f : \mathbb{Z} \to \mathcal{P}(A)$



every vertex in V

distance from A is k

$$f(k) = \{n \in V : d(n, A) = k\}$$

A set

$$P = \{f(k) \mid k \in \mathbb{Z}\}$$

$k \in \{0, 1, 2, 3, 4\}$ is!

$f(0) = \{A\}$

$f(1) = \{B, C, H\}$

$f(2) = \{D, E, I\}$

$f(3) = \{G\}$

$f(4) = \{F\}$

$f(5) = \{\}$

$f(-1) = \{\}$

P is Not a partition b/c

Partitions

Is $\{\{1,2\}, \{2,3\}\}$ a partition of $\{1, 2, 3\}$?

No overlap

Is $\{\{1\}, \{2,3\}, \emptyset\}$ a partition of $\{1, 2, 3\}$?

No, empty set

Is $\forall x \in \mathbb{Z}$, $\{\{x < 5\}, \{x \geq 5\}\}$ a partition of $\mathbb{Z}$? A partition of $\mathbb{R}$?

Yes

No!

4.5 not in $x \in \mathbb{Z}$

Partitions with Infinite Sets

A collection of sets C is a partition of A iff

(1) C covers A: $\bigcup_{x \in C} x = A$

(2) No overlap within elements of C: $X \cap Y = \emptyset$ $(\forall X \neq U, X, Y \in C)$

(3) No element of C is empty

Ex: Partitioning all real numbers that round to the same integer.

In-class Example

A: $\{2, 3, 4, 5, 10, 12\}$

Define $F : A \to \mathbb{P}(A)$ such that $F(x) = \{y \in A \mid \text{y is a factor of x}\}$

What are the values of $F(12)$? $F(5)$?

$F(12) = \{2, 3, 4, 12\}$

$F(5) = \{5\}$

Define $S = \{F(x) \mid x \in A\}$. What is S? Is S a partition of A?

$\{\{2\}, \{3\}, \{2,4\}, \{5\}, \{5,10\}, \{2,3,4,12\}\}$ No!

Overlaps ∴

Binomial Theorem

Suppose you flip a coin n times. How many ways can you get k heads?

$$\binom{n}{k}$$

Assuming a completely fair coin, what is the probability of k heads?

$$\binom{n}{k} 0.5^{n} \; 0.5^{(n-k)}$$

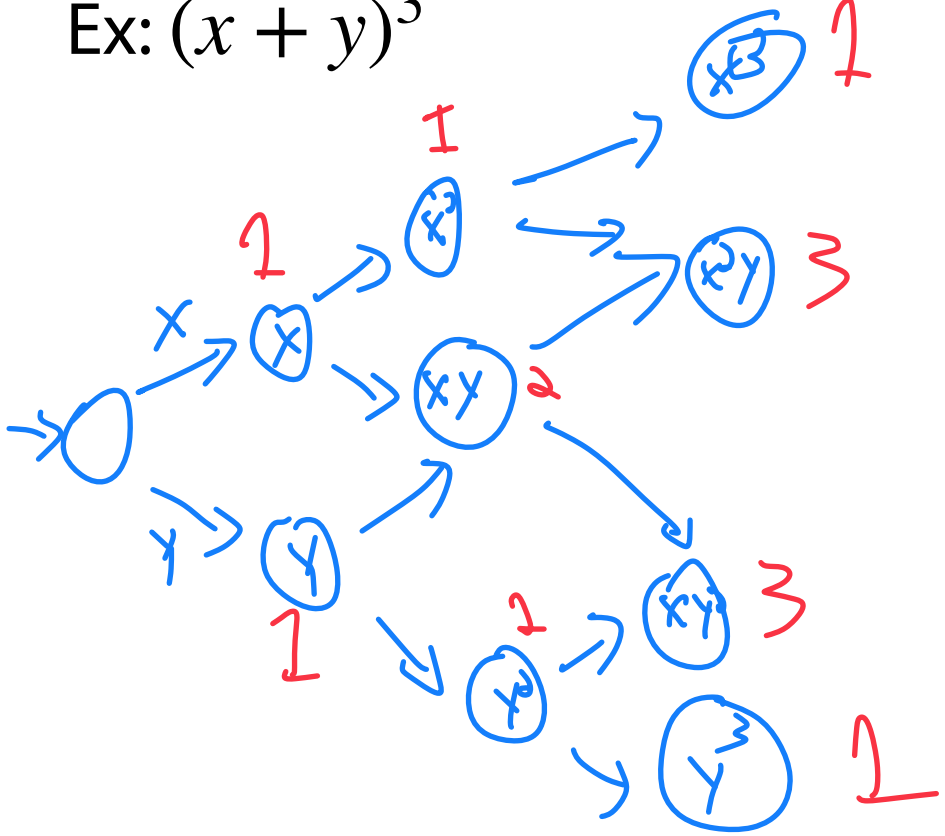
What if there is a 60% chance of heads?

$$\binom{n}{k} 0.6^{n} \; 0.4^{(n-k)}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Ex: $(x+y)^3$



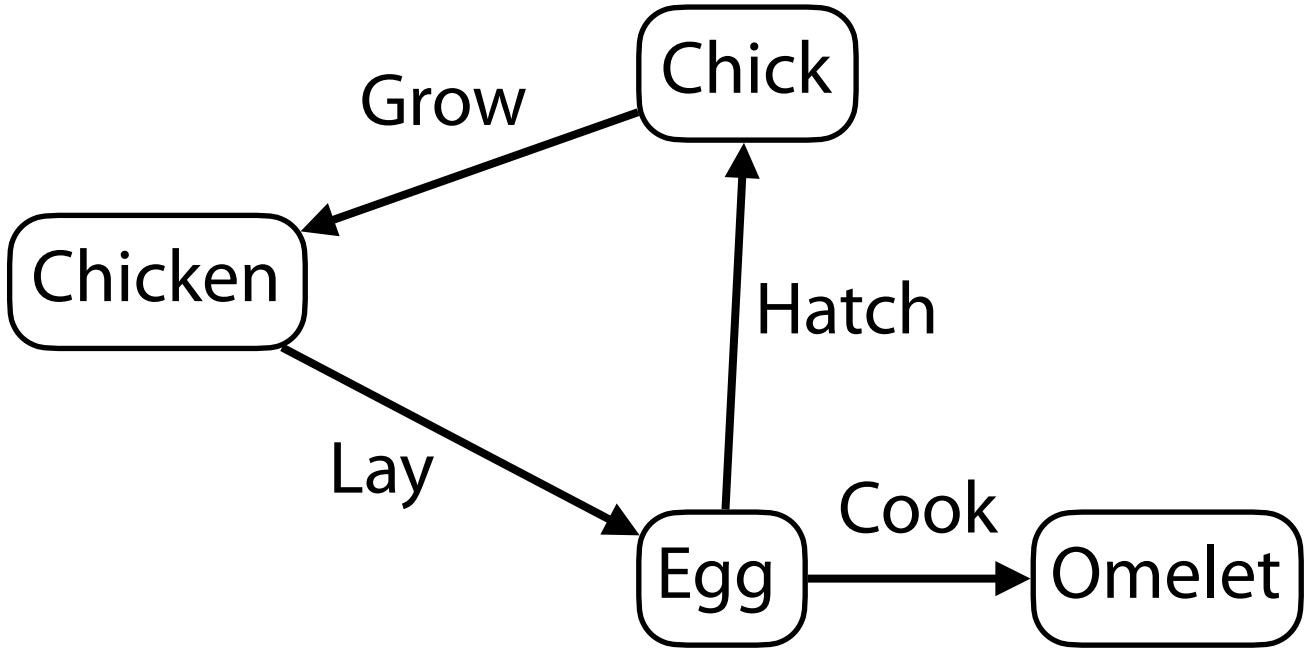
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

$$= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

$$1 \quad 3 \quad 3 \quad 1$$

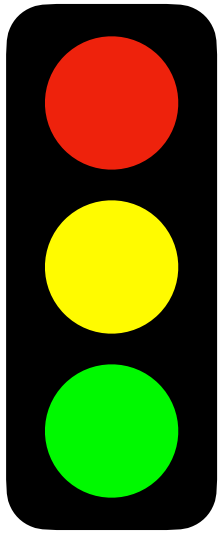
State Diagrams

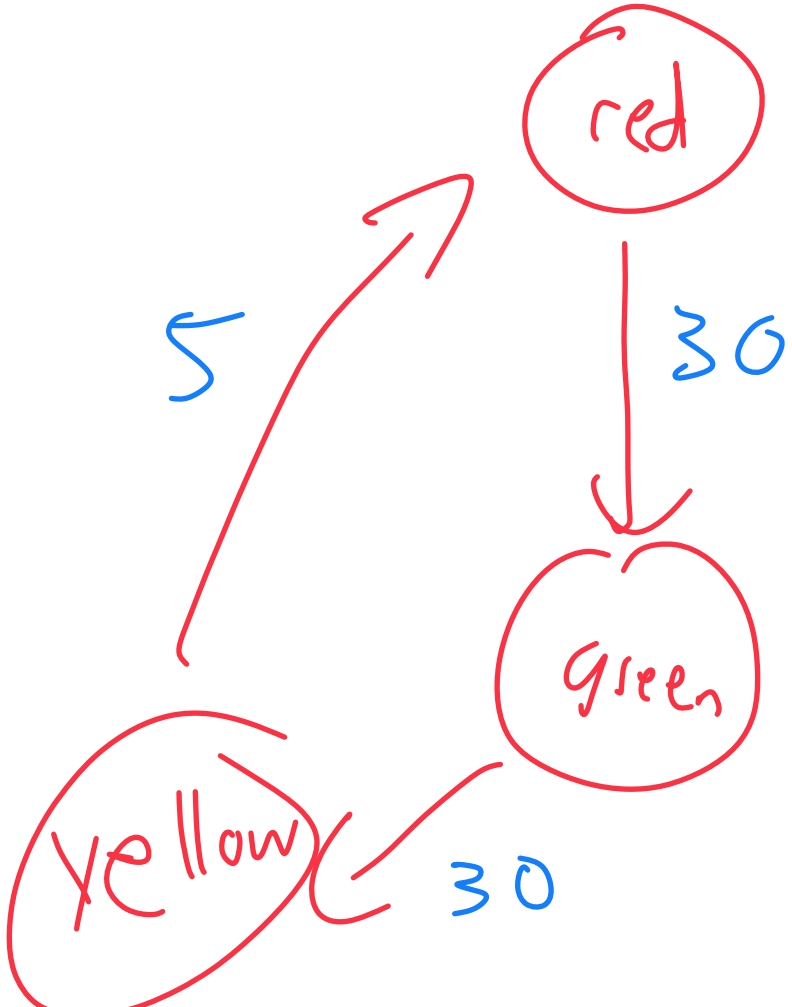
State Diagrams are directed graphs where vertices are states and edges are drawn between states when an action leads from one to the other



State Diagrams

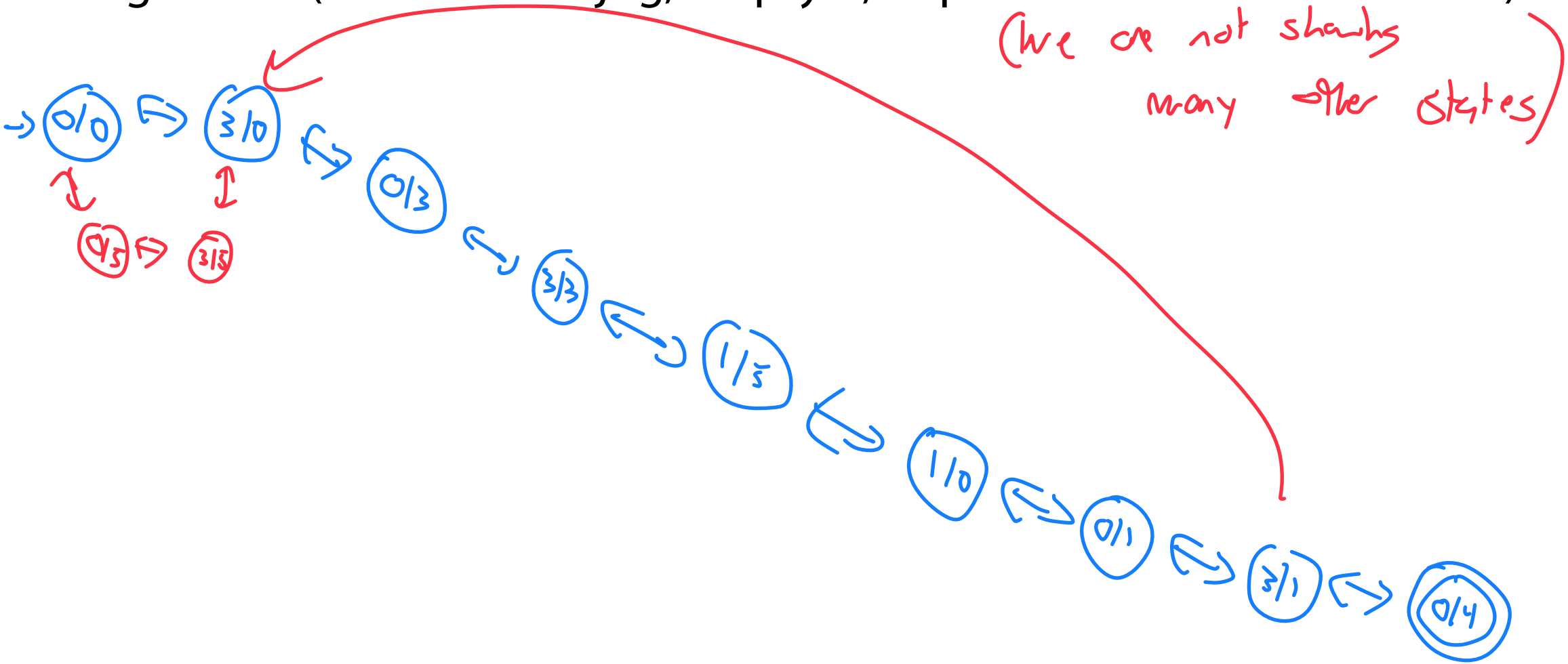
How would you describe or draw the state diagram for a traffic light?





State Diagrams: Start and End States

Given a three gallon jug and a five gallon jug, how can we measure out four gallons? (You can fill a jug, empty it, or pour from one to the other).



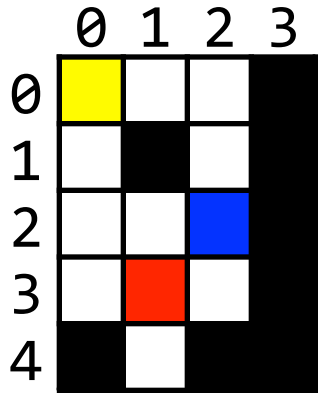
State Diagrams: Transition Function

A state diagram consists of a set of states S, a set of actions A, and a **transition function**, δ, that maps out the edges of the graph.

$\delta : S \times A \rightarrow \mathbb{P}(S)$

S : {Coordinates in maze}

A : {*Up*, *Down*, *Left*, *Right*}



function allows all state action pairs

↳ But non-valid ones are ∅

Is powerset b/c allows empty or any combo of states

1,3

, *Up* → 1,2

2,2

, *Right* → ∅

0,0

Up → ∅

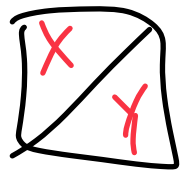
Down → (0,1)

Left → ∅

Right → (1,0)

0,0

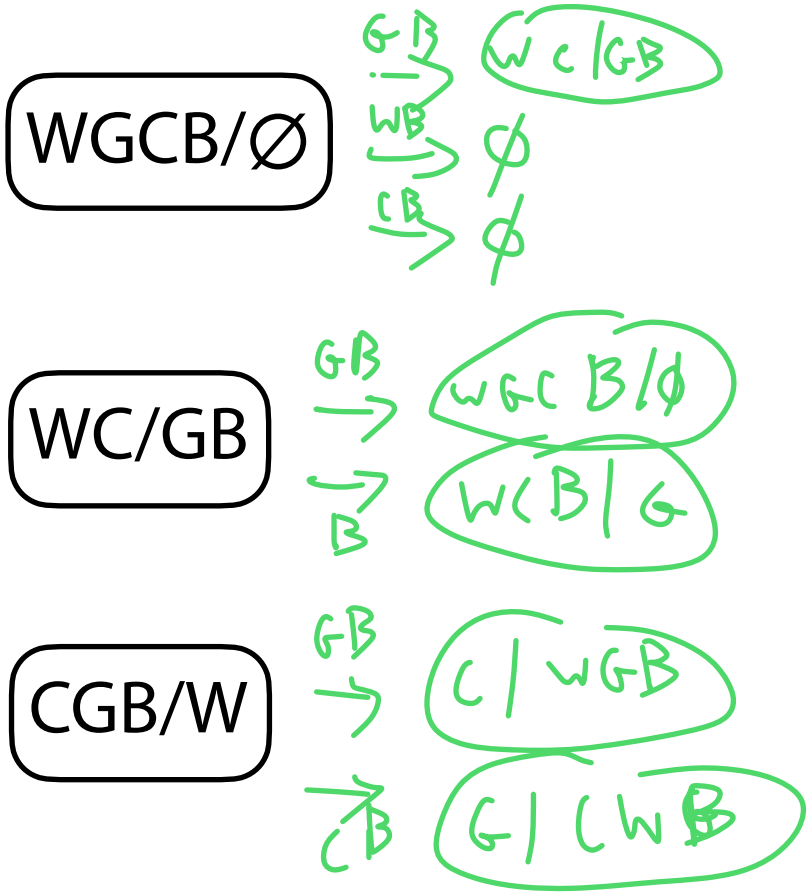
Right



((0,0), right) can have two or more outcomes!

State Diagram: Wolf-goat-cabbage

A farmer has a wolf, a goat, and a cabbage and needs to cross a river. He can only take one object across the river on a boat at a time. The goat and wolf as well as the goat and cabbage can't be alone together.



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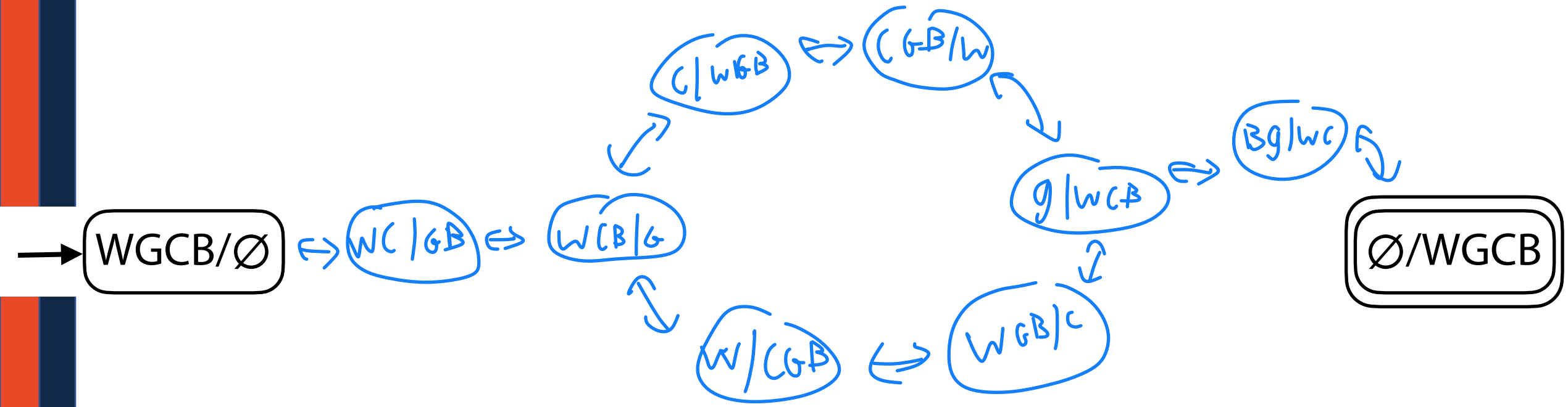


State Diagrams: Transition Function

States: Artifact, Bat, Entrance, Gold, Snake, Trap

Transitions:

(E, North)→ B

(E, East) → S

(B, South)→ E

(B, East)→ T

(S, West)→ E

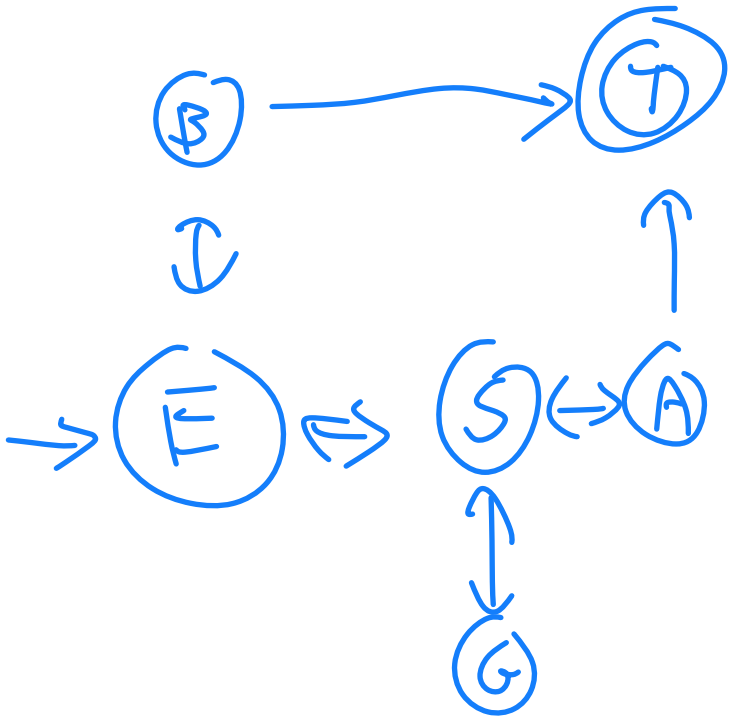
(S, South)→ G

(S, West)→ A

(A, West)→ S

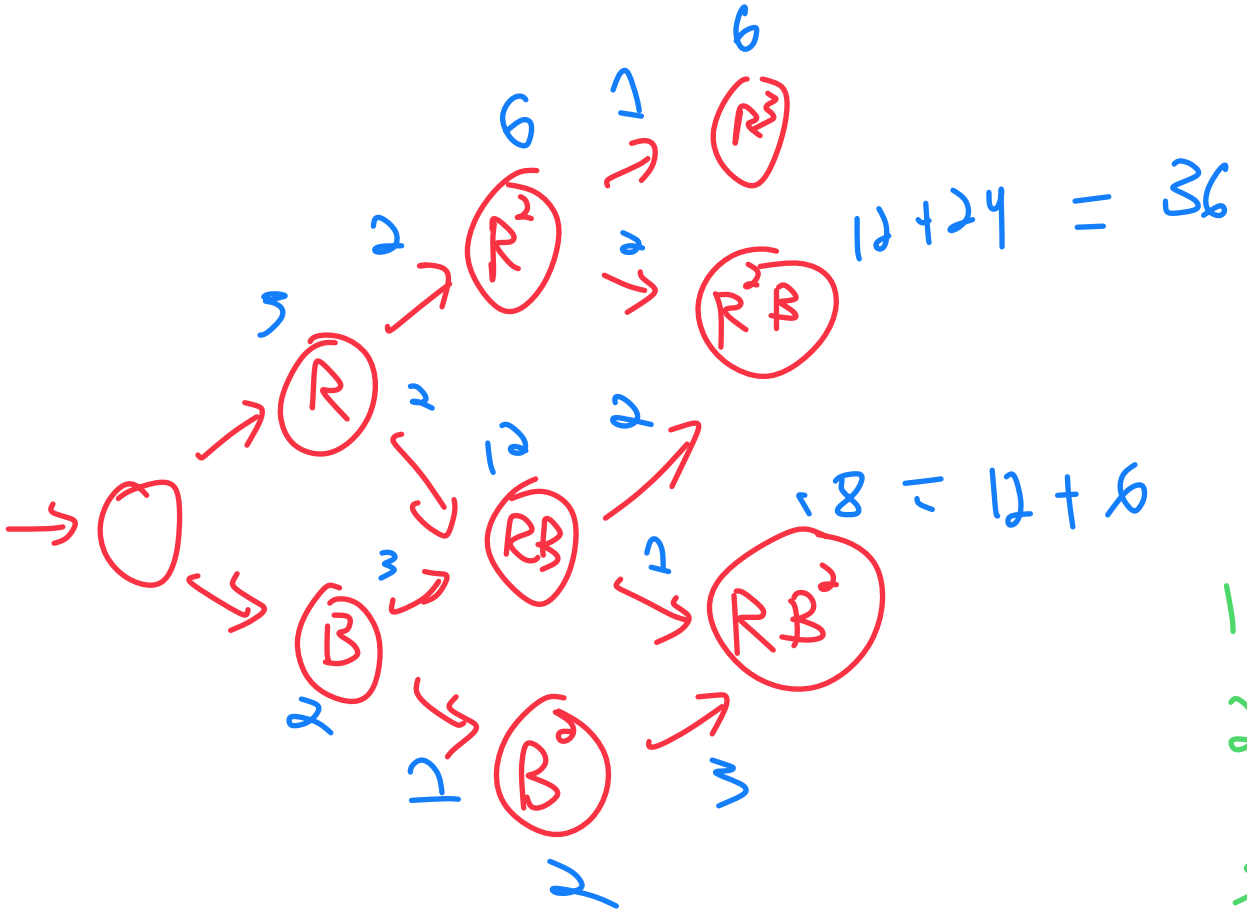
(A, North)→ T

(G, North)→ S

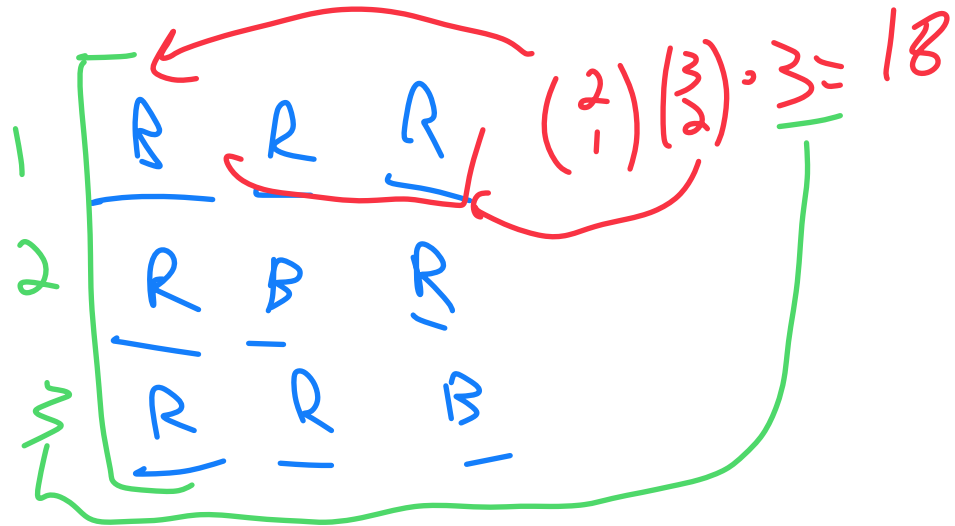


State Diagrams: Probabilities

Given a bag with three red marbles and two blue marbles, you draw three marbles. What is probability you have exactly one blue marble?



$$\frac{18}{60} = \frac{3}{10}$$



State Diagrams: Probabilities

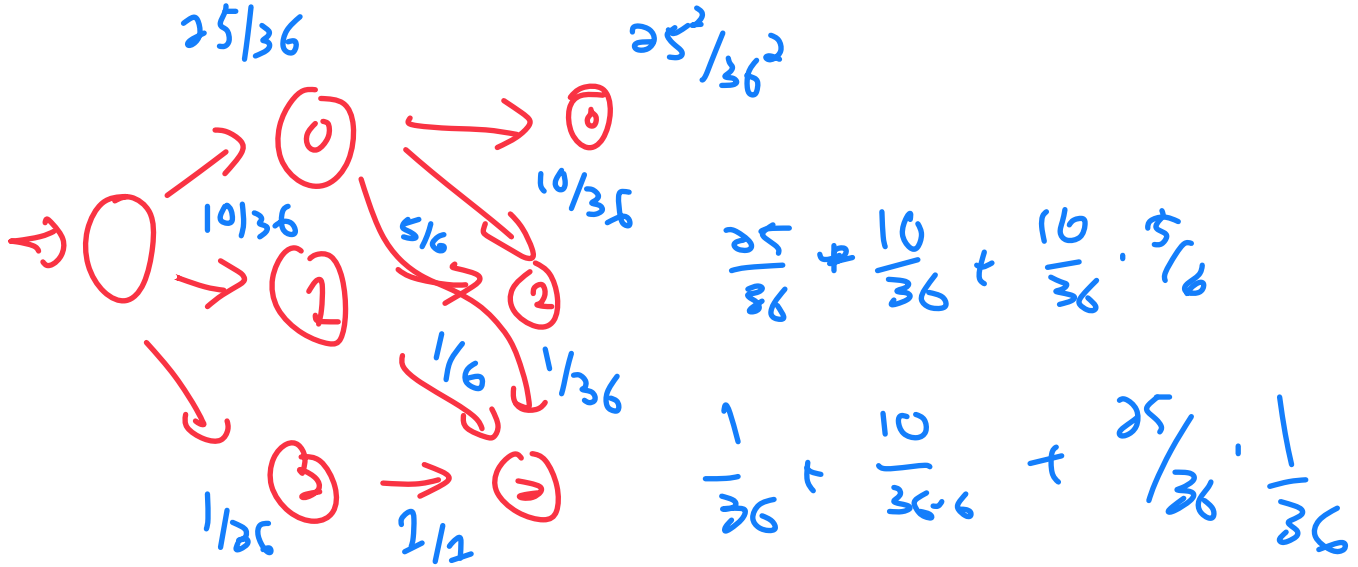
What is the chance of rolling two ones in one roll of two six-sided dice?

$\frac{1}{36}$

exactly one 1?

$\frac{1}{6} \cdot \frac{5}{6} \cdot 2 = \frac{10}{36}$

If you are allowed to re-roll, what is the probability of getting it?



$\frac{25}{36} + \frac{10}{36} + \frac{10}{36} \cdot \frac{5}{6}$

$\frac{1}{36} + \frac{10}{36 \cdot 6} + \frac{25}{36} \cdot \frac{1}{36}$