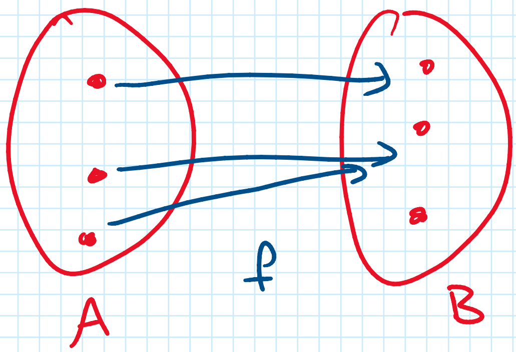
CS 173 Lecture 7c: One-to-one functions



$f: A \to B$ is one-to-one if no two elements of the domain have the same image.

i.e. $\forall x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

i.e. $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$ ← almost always use this in proofs.

negation: $\exists x, y \in A$, $f(x) = f(y)$ & $x \neq y$.

Claim: $f: \mathbb{Z} \to \mathbb{Z}$ $f(x) = \lfloor x/2 \rfloor$ is not one-to-one.

Proof: Let $x = 0$ & $y = 1$.

Then $x \neq y$, but $f(x) = \lfloor 0/2 \rfloor = 0 = \lfloor 1/2 \rfloor = f(y)$.

So f is not one-to-one. □

Claim: Suppose $g: \mathbb{Z} \to \mathbb{Z}$ is one-to-one.

Then $f: \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ $f(x) = (g(x)|x|, |x|)$ is also one-to-one.

Proof. Let x, y be integers such that $f(x) = f(y)$. Goal: $x = y$

That means $(g(x)|x|, |x|) = (g(y)|y|, |y|)$.

i.e., $g(x)|x| = g(y)|y|$ & $|x| = |y|$.

Case 1: $x = 0$. Then $|x| = 0 = |y|$.

Then $y = 0$ as well. So $x = y$

Case 1: $x=0$. Then $|x|=0=|y|$.

Then $y=0$ as well, so $x=y$.

Case 2: $x\neq 0$. Then $|x|=|y|\neq 0$.

Since $g(x)|x| = g(y)|y| = g(y)|x|$, and $|x|\neq 0$, we can divide by $|x|$ to get $g(x)=g(y)$.

Since g is one-to-one, $g(x)=g(y)$ implies $x=y$.

This covers all possible values of x, and in both cases $f(x)=f(y) \Rightarrow x=y$.

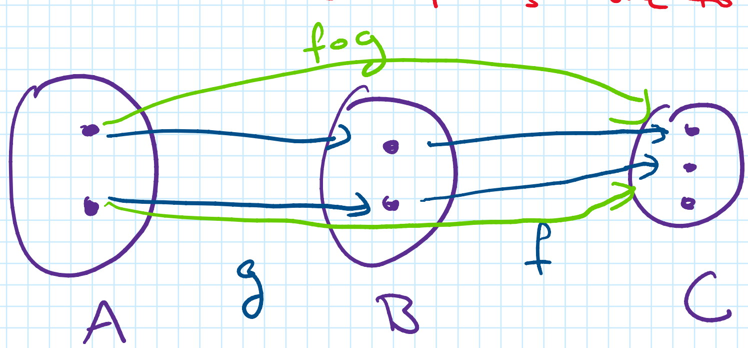
So f is one-to-one. $\square$

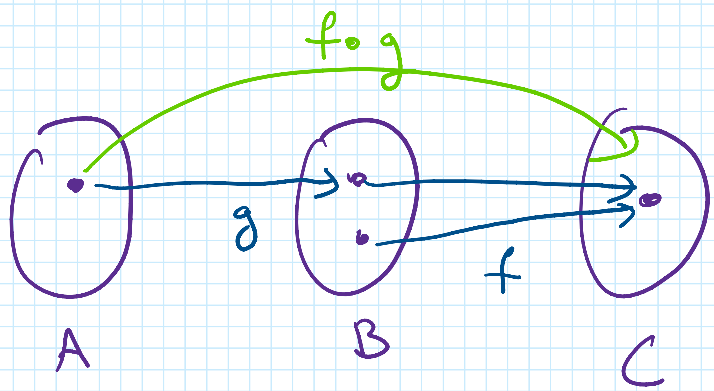
Claim: Suppose $f: B\to C$ & $g: A\to B$ have the following properties:

g is onto & $f\circ g: A\to C$ is one-to-one.

$f(g(x))\in C$, $x\in A$

Then f is one-to-one.





goal: $x=y$

A B C

$x = y$

Proof. Let $x, y \in B$ and suppose that $f(x) = f(y)$

Since g is onto, there exist $p, q \in A$ such that $g(p) = x$ & $g(q) = y$.

Since $f(x) = f(y)$, $g(p) = x$, & $g(q) = y$,

then $f(g(p)) = f(g(q))$

Since $f \circ g$ is one-to-one,

$p = q$.

That means $g(p) = g(q)$, i.e. $x = y$.

So f is injective. $\square$